

NEIGHBORS OF SEIFERT SURGERIES ON A TREFOIL KNOT IN THE SEIFERT SURGERY NETWORK

ARNAUD DERUELLE, KATURA MIYAZAKI, AND KIMIHIKO MOTEGI

Dedicated to Fico González Acuña on the occasion of his 70th birthday

ABSTRACT. A Seifert surgery is a pair (K, m) of a knot K in S^3 and an integer m such that m -Dehn surgery on K results in a Seifert fiber space allowed to contain fibers of index zero. Twisting K along a trivial knot called a seiferter for (K, m) yields Seifert surgeries. We study Seifert surgeries obtained from those on a trefoil knot by twisting along their seiferters. Although Seifert surgeries on a trefoil knot are the most basic ones, this family is rich in variety. For any $m \neq -2$ it contains a successive triple of Seifert surgeries (K, m) , $(K, m+1)$, $(K, m+2)$ on a hyperbolic knot K , e.g. 17-, 18-, 19-surgeries on the $(-2, 3, 7)$ pretzel knot. It contains infinitely many Seifert surgeries on strongly invertible hyperbolic knots none of which arises from the primitive/Seifert-fibered construction, e.g. (-1) -surgery on the $(3, -3, -3)$ pretzel knot.

1. INTRODUCTION

A pair (K, m) of a knot K in S^3 and an integer m is a *Seifert surgery* if the result $K(m)$ of m -Dehn surgery is a Seifert fiber space which may contain a fiber of index 0, i.e. a degenerate fiber. In this paper we allow Seifert fibrations to contain degenerate fibers. If $K(m)$ admits a degenerate Seifert fibration, it is either a lens space or a connected sum of two lens spaces [5, Proposition 2.8(2), (3)]. For a Seifert surgery (K, m) , when $K(m)$ admits a non-degenerate Seifert fibration (i.e. $K(m)$ is not a connected sum of two lens spaces), to emphasize this fact we also say that (K, m) is a *Seifert fibered surgery*.

In [5], we relate Seifert surgeries by twists along “seiferters” and define a 1-dimensional complex called the Seifert Surgery Network. We briefly review the definition of the network. Let (K, m) be a Seifert surgery. A knot $c \subset S^3 - K$ is a *seiferter* for (K, m) if c is a trivial knot in S^3 but becomes a fiber in a Seifert fibration of $K(m)$. Let K_p and m_p be the images of K and m under twisting p times along c ; in fact, $m_p = m + p(\text{lk}(K, c))^2$. Then, (K_p, m_p) remains a Seifert surgery, and the image of c under the twisting is a seiferter for (K_p, m_p) ; see the commutative diagram below. We also consider twists along an “annular pair of seiferters”. For two seiferters c_1, c_2 for (K, m) , if c_1 and c_2 are fibers in the same Seifert fibration of $K(m)$, then the (unordered) pair $\{c_1, c_2\}$ is a *pair of seiferters*. A pair of seiferters is an *annular pair of seiferters* if c_1 and c_2 cobound an annulus A in S^3 . After twisting along the annulus A the images of (K, m) and $\{c_1, c_2\}$

2010 *Mathematics Subject Classification.* Primary 57M25, 57M50 Secondary 57N10

Key words and phrases. Dehn surgery, hyperbolic knot, Seifert fiber space, seiferter, trefoil knot

remain a Seifert surgery and an annular pair of seiferters for it. The vertices of the *Seifert Surgery Network* are all Seifert surgeries, and two vertices of the network are connected by an edge if one vertex (Seifert surgery) is obtained from the other by a single twist along a seifert or an annulus cobounded by an annular pair of seiferters. Refer to [5, Subsection 2.4] for details of the definition.

$$\begin{array}{ccc}
 (K, m) & \xrightarrow{\text{twist along } c \text{ (resp. } \{c_1, c_2\})} & (K_p, m_p) \\
 \downarrow m\text{-surgery on } K & & \downarrow m_p\text{-surgery on } K_p \\
 K(m) & \xrightarrow{\text{surgery along } c \text{ (resp. } c_1 \cup c_2)} & K_p(m_p)
 \end{array}$$

DIAGRAM 1.

- Remark 1.1.* (1) In [5], an annular pair $\{c_1, c_2\}$ is defined to be an ordered pair of c_1 and c_2 to specify the direction of twist along the annulus cobounded by $c_1 \cup c_2$. However, since we do not perform annulus twists in this paper, annular pairs are presented as unordered pairs.
- (2) If a seifert c for (K, m) bounds a disk in $S^3 - K$, we call c *irrelevant* and do not regard it as a seifert. This is because no twists along irrelevant seiferters change Seifert surgeries. However, for pairs of seiferters $\{c_1, c_2\}$ we allow c_i to be irrelevant. Let $\{c_1, c_2\}$ be an annular pair for (K, m) . If either c_1 and c_2 cobound an annulus disjoint from K or there is a 2-sphere in S^3 separating c_i and $c_j \cup K$, then twists along $\{c_1, c_2\}$ do not change (K, m) or have the same effect on K as twists along c_j . We thus call such an annular pair *irrelevant*, and exclude it from annular pairs of seiferters.

Any integral surgery on a torus knot $T_{p,q}$ ($|p| > q \geq 1$) has at least three seiferters. Let s_p and s_q be the exceptional fibers of indices $|p|$ and q in the Seifert fibration of the exterior of $T_{p,q}$, respectively; see Figure 1.1. Since the Seifert fibration of the exterior extends to $T_{p,q}(m)$ for any integer m , the trivial knots s_p, s_q are seiferters for $(T_{p,q}, m)$. Furthermore, a meridian c_μ of $T_{p,q}$ is also a seifert for $T_{p,q}(m)$ because c_μ is isotopic to the core of the filled solid torus in $T_{p,q}(m)$. The seiferters s_p, s_q, c_μ are fibers of indices $|p|, q, |pq - m|$ in $T_{p,q}(m)$, respectively. We call them *basic seiferters* for $T_{p,q}$. Note that s_p, s_q, c_μ in Figure 1.1 are fibers in a Seifert fibration of $T_{p,q}(m)$, simultaneously, and any two of these seiferters cobound an annulus in S^3 . Thus, $\{s_p, s_q\}, \{s_p, c_\mu\}, \{s_q, c_\mu\}$ in Figure 1.1 are annular pairs of seiferters for $(T_{p,q}, m)$, called *basic annular pairs*.

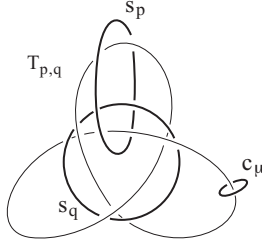


FIGURE 1.1. Basic seiferters for $T_{p,q}$, where $p = -3, q = 2$

In the network, a path from $(T_{p,q}, m)$ to (K, m') tells how the Seifert surgery (K, m') is obtained from $(T_{p,q}, m)$ by a sequence of twistings along seiferters and/or annular pairs of seiferters. However, we cannot obtain a non-torus knot by twisting a torus knot along its basic seiferters or basic annular pairs. To obtain a Seifert surgery on a hyperbolic knot we need to twist along a “hyperbolic seiferters”. A seiferters c (resp. an annular pair $\{c_1, c_2\}$) for (K, m) is *hyperbolic* if $S^3 - K \cup c$ (resp. $S^3 - K \cup c_1 \cup c_2$) admits a complete hyperbolic metric of finite volume. Twists along a “hyperbolic seiferters” or a “hyperbolic annular pair” yield infinitely many Seifert surgeries on hyperbolic knots. We denote by $\mathcal{N}(T_{-3,2})$ the set of Seifert surgeries obtained from $(T_{-3,2}, m)$ ($m \in \mathbb{Z}$) by twisting arbitrary times along seiferters or annular pairs for $(T_{-3,2}, m)$.

In this paper, we find hyperbolic seiferters and hyperbolic annular pairs of seiferters for $(T_{-3,2}, m)$, and study Seifert surgeries on hyperbolic knots that belong to $\mathcal{N}(T_{-3,2})$. We construct hyperbolic seiferters (resp. hyperbolic annular pairs) for $(T_{-3,2}, m)$ by applying “ m -moves” to basic seiferters (resp. basic annular pairs) for $T_{-3,2}$. An m -move is, in fact, a Kirby calculus handle-slide over an m -framed knot, and the definition is given in Section 2. Theorem 1.2 below follows from Corollaries 3.8, 4.9.

Theorem 1.2. *A Seifert surgery $(T_{-3,2}, m)$ has a hyperbolic seiferters for any integer $m \neq -4$; $(T_{-3,2}, -4)$ has at least six hyperbolic annular pairs of seiferters. Furthermore, if $m \leq -8$ or $-1 \leq m$, then $(T_{-3,2}, m)$ has at least three hyperbolic seiferters and nine hyperbolic annular pairs of seiferters.*

The following two theorems are about surgeries belonging to $\mathcal{N}(T_{-3,2})$. A small Seifert fiber space is a 3-manifold which admits a non-degenerate Seifert fibration over the 2-sphere containing exactly three exceptional fibers. We call a Seifert surgery (K, m) a *small Seifert fibered surgery* if $K(m)$ is a small Seifert fiber space.

Theorem 1.3. *For any integer m , there is a hyperbolic knot whose m -, $(m+1)$ -, $(m+2)$ -surgeries are small Seifert fibered surgeries. If $m \neq -2$, then such three successive surgeries can be found in $\mathcal{N}(T_{-3,2})$ (Theorem 3.13).*

Theorem 1.4. *The neighborhood $\mathcal{N}(T_{-3,2})$ contains infinitely many small Seifert fibered surgeries on strongly invertible hyperbolic knots which do not arise from the primitive/Seifert-fibered construction introduced in [4] (Theorem 5.4).*

Figure 1.2 is a portion of the subnetwork $\mathcal{N}(T_{-3,2})$. Twists along the meridian c_μ generate the horizontal line in Figure 1.2, which contains all integral surgeries on $T = T_{-3,2}$. The trivial knots c^m ($m = -1, -6$) in Figure 1.2 are hyperbolic seiferters for $(T_{-3,2}, m)$, $(T_{-3,2}, m-1)$, $(T_{-3,2}, m-2)$ (Corollary 3.3, Proposition 3.7); c in Figure 1.2 is a hyperbolic seiferters for $(T_{-3,2}, -1)$ (Lemma 5.3). Note that in Figure 1.2 the images of seiferters under twisting are denoted by the same symbols as originals. The (-2) -twist on $T_{-3,2}$ along c^{-1} yields the figure-eight knot K . Thus (-2) -twist along c^{-1} converts $(T_{-3,2}, m)$ to $(K, m-2w^2)$, where $m = -1, -2, -3$ and $w = \text{lk}(T_{-3,2}, c^{-1}) = 0$. The 1-twist along c^{-6} yields the $(-2, 3, 7)$ pretzel knot P , so that the 1-twist along c^{-6} converts $(T_{-3,2}, m)$ to $(P, m+w^2)$, where $m = -6, -7, -8$ and $w = \text{lk}(T_{-3,2}, c^{-6}) = 5$. The slanted line through $(T_{-3,2}, -1)$ is generated by twists along the seiferters c . The 1-twist along c yields (-1) -surgery on the $(3, -3, -3)$ pretzel knot. All Seifert surgeries on the slanted line except $(T_{-3,2}, -1)$ do not arise from the primitive/Seifert-fibered construction.

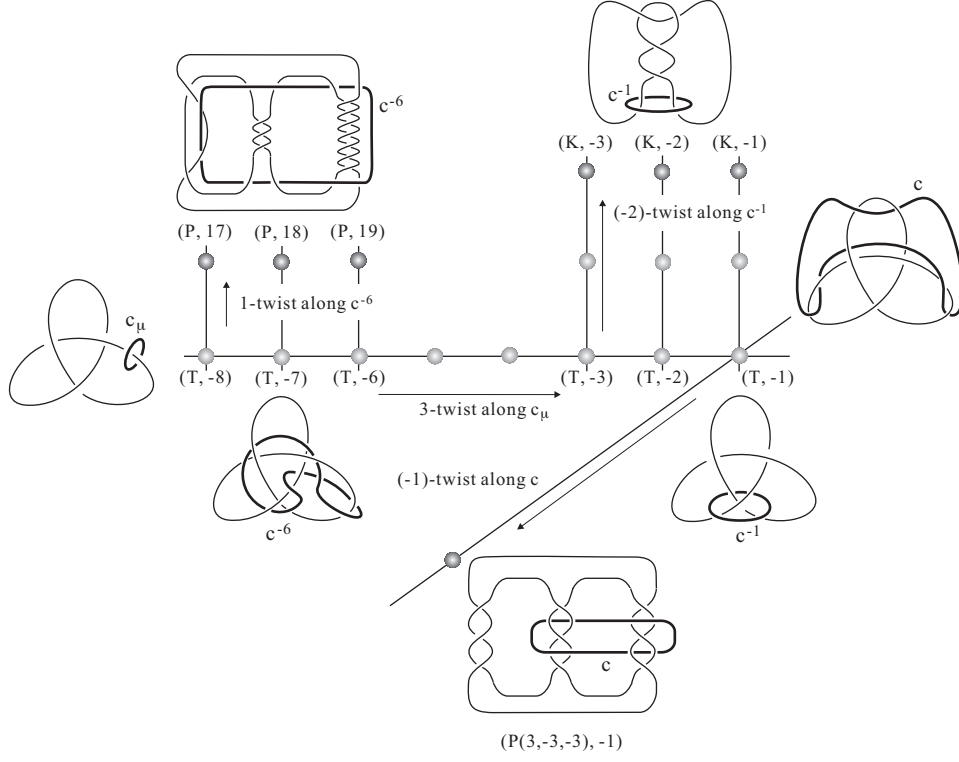


FIGURE 1.2. $T = T_{-3,2}$, K is the figure-eight knot, and P is the $(-2, 3, 7)$ pretzel knot.

2. PRELIMINARIES

In this section we recall some results on m -moves to seiferters and annular pairs.

Definition 2.1 (m -moves). Let K be a knot in S^3 with a tubular neighborhood $N(K)$, and c a knot in $S^3 - N(K)$. Take a simple closed curve α_m on $\partial N(K)$ representing a slope m . Let b be a band in $S^3 - \text{int}N(K)$ connecting α_m and c , and let $b \cap \alpha_m = \tau_{\alpha_m}$, $b \cap c = \tau_c$. We set $\tau'_{\alpha_m} = \alpha_m - \text{int}\tau_{\alpha_m}$ and $\tau'_c = c - \text{int}\tau_c$. Then the band connected sum $c \natural_b \alpha_m = \tau'_c \cup (\partial b - \text{int}(\tau_c \cup \tau_{\alpha_m})) \cup \tau'_{\alpha_m}$ is a knot in $S^3 - \text{int}N(K)$. Pushing $c \natural_b \alpha_m$ away from $\partial N(K)$, we obtain a knot c' in $S^3 - N(K)$; see Figure 2.1. We say that c' is obtained from c by an m -move using the band b .

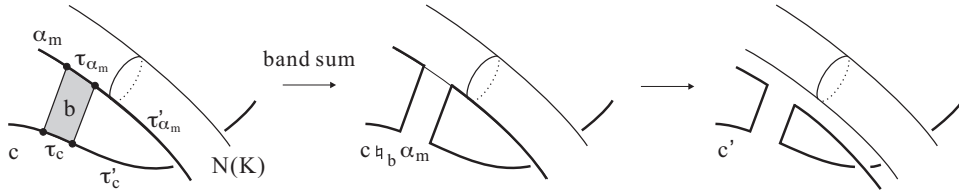


FIGURE 2.1. m -move

Proposition 2.2 ([5, Propositions 2.19(3), 2.22]). *Let K be a nontrivial knot in S^3 . Suppose that (K, m) is a Seifert surgery with a seifert c .*

- (1) *Assume that c' is obtained from c by a finite sequence of m -moves. Then, c' is isotopic to c in $K(m)$. Moreover, if c' is unknotted in S^3 , then c' is also a seifert for (K, m) .*
- (2) *If c' is obtained from c by a single m -move and has an orientation induced from c , then $\text{lk}(K, c') = \text{lk}(K, c) + \varepsilon m$ where $\varepsilon = \pm 1$.*
- (3) *Assume that c' is obtained from c by a single m -move. We give K and α_m parallel orientations, c' an orientation induced from α_m , and c an orientation induced from c' . Then, an n -framing of c becomes an $(n + 2\text{lk}(K, c) + m)$ -framing of c' after an isotopy in $K(m)$.*

We generalize m -moves to pairs of seiferters.

Definition 2.3 (m -moves to pairs). Let $c_1 \cup c_2$ be a link in $S^3 - N(K)$. Let $\alpha_m \subset \partial N(K)$ be a simple closed curve representing a slope m , and b a band connecting c_1 and α_m with $b \cap c_2 = \emptyset$. Isotoping the band sum $c_1 \natural_b \alpha_m (\subset S^3 - \text{int} N(K))$ away from $\partial N(K)$ without meeting c_2 , we obtain a knot $c'_1 \subset S^3 - N(K)$. Then we say that the link $c'_1 \cup c_2$ is obtained from $c_1 \cup c_2$ by an m -move using the band b .

Proposition 2.4 ([5, Proposition 2.25]). *Let K be a knot in S^3 , and m a slope on $\partial N(K)$. Let $c_1 \cup c_2$ and $c'_1 \cup c_2$ be links in $S^3 - N(K)$ with each component trivial in S^3 . Suppose that $c'_1 \cup c_2$ is obtained from $c_1 \cup c_2$ by an m -move. Then we have:*

- (1) *The two ordered links $c_1 \cup c_2$ and $c'_1 \cup c_2$ are isotopic in $K(m)$.*
- (2) *If $\{c_1, c_2\}$ is a pair of seiferters for (K, m) , then $\{c'_1, c_2\}$ is also a pair of seiferters for (K, m) .*

Corollary 2.5 ([5, Proposition 2.26]). *Let $c_1 \cup c_2$ and $c'_1 \cup c'_2$ be links in $S^3 - N(K)$ with each component trivial in S^3 . Let $\alpha_i \subset \partial N(K)$ be a simple closed curve with slope m , and b_i a band connecting c_i and α_i such that $(c_1 \cup b_1 \cup \alpha_1) \cap (c_2 \cup b_2 \cup \alpha_2) = \emptyset$. Suppose that $c'_1 \cup c'_2$ is obtained from $(c_1 \natural_{b_1} \alpha_1) \cup (c_2 \natural_{b_2} \alpha_2)$ by an isotopy in $S^3 - \text{int} N(K)$. Then, $\{c'_1, c'_2\}$ is a pair of seiferters for (K, m) if $\{c_1, c_2\}$ is a pair of seiferters for (K, m) .*

The following proposition will be used to show that a pair of seiferters for (K, m) does not cobound an annulus disjoint from K .

Proposition 2.6 ([5, Proposition 2.36]). *Let c_1 and c_2 be possibly irrelevant seiferters for (K, m) with respect to a Seifert fibration \mathcal{F} of $K(m)$. Suppose that c_1 and c_2 cobound an annulus A in $S^3 - \text{int} N(K)$. Then the following hold.*

- (1) $\text{lk}(c_1, K) = \text{lk}(c_2, K)$.
- (2) *If $K(m)$ is not a lens space, then c_1 and c_2 are regular fibers in \mathcal{F} . If $K(m)$ is a lens space, then we have a Seifert fibration (possibly distinct from \mathcal{F}) having c_1 and c_2 as regular fibers.*

3. SEIFERTERS FOR SEIFERT SURGERIES ON A TREFOIL KNOT

It is known that there is a hyperbolic knot which admits Seifert fibered surgeries for three successive (integral) surgery slopes. Well-known examples are the $(-1)-$, $(-2)-$, $(-3)-$ surgeries on twist knots [2], and the $17-$, $18-$, $19-$ surgeries on the $(-2, 3, 7)$ pretzel knot [8]. In this section, we show that for any integer $m \notin \{-5, -4, -3, -2\}$, the $m-$, $(m-1)-$, $(m-2)-$ surgeries on the trefoil knot

$T_{-3,2}$ have a hyperbolic seifert in common. Then, arbitrary twists on $T_{-3,2}$ along the seifert produce a knot with three successive Seifert surgeries. We show that the three successive Seifert fibered surgeries on twist knots and the $(-2, 3, 7)$ pretzel knot arise in this manner.

Let $\alpha_m \subset \partial N(T_{-3,2})$ be a simple closed curve representing a slope $m \in \mathbb{Z}$. Let b_μ, b_{-3}, b_2 be the band in $S^3 - \text{int}N(T_{-3,2})$ connecting α and c_μ, s_{-3}, s_2 , respectively as described in Figure 3.1. We denote by c_1^m, c_2^m, c_3^m the knots obtained from c_μ, s_{-3}, s_2 by single m -moves via these bands, respectively.

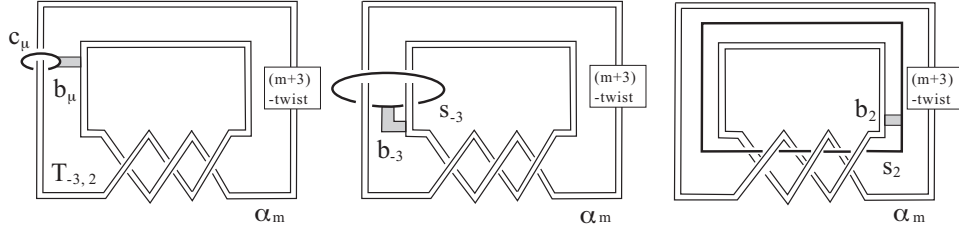


FIGURE 3.1. Band sums $c_\mu \natural_{b_\mu} \alpha_m, s_{-3} \natural_{b_{-3}} \alpha_m, s_2 \natural_{b_2} \alpha_m$

Lemma 3.1. *For any integer m and $i \in \{1, 2, 3\}$, c_i^m satisfy the following.*

- (1) *The knots c_1^m, c_2^m, c_3^m are isotopic in $T_{-3,2}(m)$ to the basic seiferters c_μ, s_{-3}, s_2 for $T_{-3,2}$, respectively. These knots are mutually distinct seiferters for $(T_{-3,2}, m)$.*
- (2) *The links $T_{-3,2} \cup c_1^m, T_{-3,2} \cup c_2^{m-1}$, and $T_{-3,2} \cup c_3^{m-2}$ are mutually isotopic in S^3 .*

Remark 3.2. Theorem 1.5 in [13] shows that if a band sum of $T_{-3,2}$ and its meridian yields a trivial knot, then such a band is unique up to isotopy. We thus see that c_1^m is the only seifert for $(T_{-3,2}, m)$ obtained from c_μ by an m -move.

Proof of Lemma 3.1. (1) The isotopies in Figures 3.2, 3.3, and 3.4 show that c_i^m ($i = 1, 2, 3$) are trivial knots in S^3 . Proposition 2.2(1) shows that c_1^m, c_2^m, c_3^m are isotopic to c_μ, s_{-3}, s_2 , respectively, and seiferters for $(T_{-3,2}, m)$.

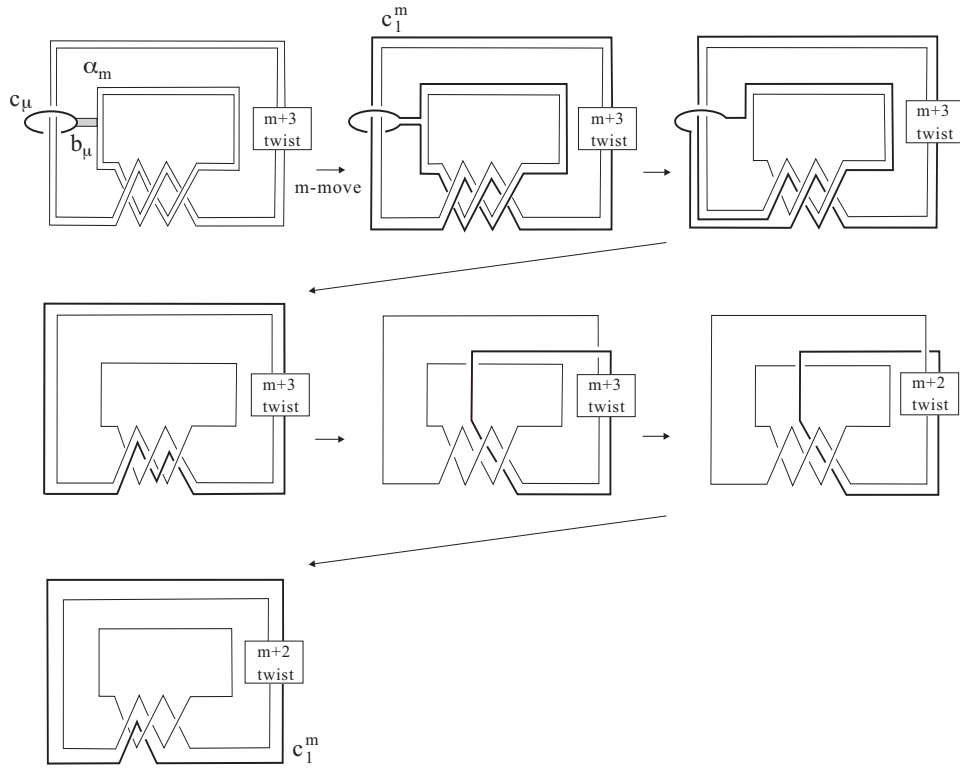
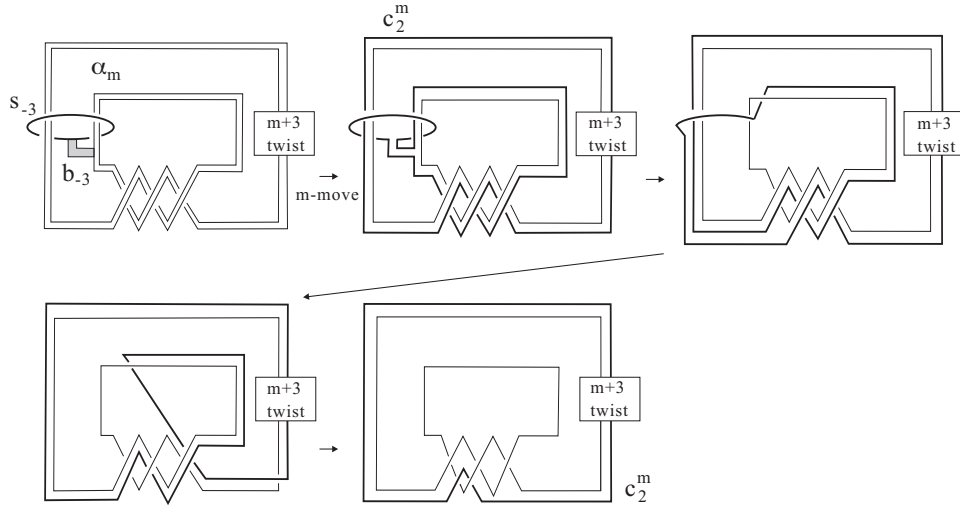
We see from Figures 3.2, 3.3, 3.4 that $|\text{lk}(T_{-3,2}, c_i^m)| = |m + i|$ ($i = 1, 2, 3$). These values are mutually distinct except when $m = -2$. If $m = -2$, then $|m + i|$ ($i = 1, 3$) are equal. However, c_1^{-2} and c_3^{-2} are isotopic in $T_{-3,2}(-2)$ to c_μ and s_2 , respectively. Since c_μ and s_2 are exceptional fibers of distinct indices 4 and 2 in $T_{-3,2}(-2)$, they are not isotopic in $T_{-3,2}(-2)$. It follows that c_i^m ($i = 1, 2, 3$) are three distinct seiferters for $(T_{-3,2}, m)$ for any m . This proves (1).

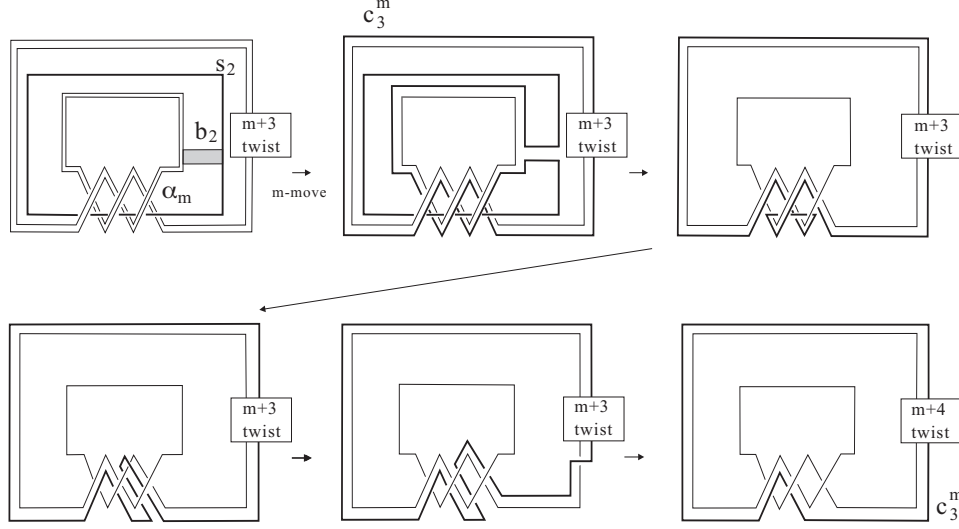
(2) Replace m with $m - 1$ (resp. $m - 2$) in Figure 3.3 (resp. Figure 3.4), we see that $T_{-3,2} \cup c_2^{m-1}$ (resp. $T_{-3,2} \cup c_3^{m-2}$) is the same link as $T_{-3,2} \cup c_1^m$. \square (Lemma 3.1)

Following Lemma 3.1(2), we denote the seiferters $c_1^m, c_2^{m-1}, c_3^{m-2}$ by c^m . Then Lemma 3.1 is rephrased as follows.

Corollary 3.3. *The knot c^m in $S^3 - T_{-3,2}$ satisfies the following.*

- (1) *c^m is the seifert c_1^m for $(T_{-3,2}, m)$, c_2^{m-1} for $(T_{-3,2}, m - 1)$, and c_3^{m-2} for $(T_{-3,2}, m - 2)$.*

FIGURE 3.2. $T_{-3,2} \cup c_1^m$ FIGURE 3.3. $T_{-3,2} \cup c_2^m$

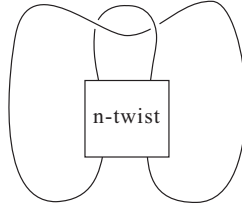
FIGURE 3.4. $T_{-3,2} \cup c_3^m$

- (2) $c^m (= c_1^m)$, $c^{m+1} (= c_2^m)$ and $c^{m+2} (= c_3^m)$ are mutually distinct seiferters for $(T_{-3,2}, m)$.

Since $|\text{lk}(T_{-3,2}, c^m)| = |m+1|$, by twisting $(T_{-3,2}, m+1-i)$ ($i = 1, 2, 3$) n times along $c^m = c_i^{m+1-i}$ we obtain Proposition 3.4 below. We denote by K_n^m the image of $T_{-3,2}$ under n -twist along c^m . As usual, we continue to denote the image of c^m after twisting along c^m by the same symbol c^m .

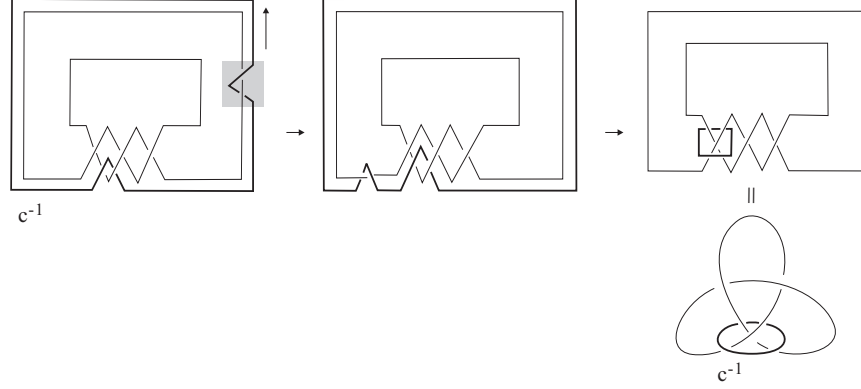
Proposition 3.4. *Let m and n be arbitrary integers. Then $(K_n^m, m+1-i+n(m+1)^2)$, where $i = 1, 2, 3$, are Seifert surgeries for which c^m is a seiferters.*

Seifert surgeries given in Proposition 3.4 contain three successive Seifert fibered surgeries on twist knots $Tw(n)$ of Figure 3.5, and the $(-2, 3, 7)$ pretzel knot $P(-2, 3, 7)$.

FIGURE 3.5. Twist knot $Tw(n)$

- Proposition 3.5.** (1) $(n-1)$ -twist along the seiferters c^{-1} converts $(T_{-3,2}, -1)$, $(T_{-3,2}, -2)$, $(T_{-3,2}, -3)$ to $(Tw(n), -1)$, $(Tw(n), -2)$, and $(Tw(n), -3)$, respectively.
- (2) 1-twist along the seiferters c^{-6} converts $(T_{-3,2}, -6)$, $(T_{-3,2}, -7)$, $(T_{-3,2}, -8)$ to $(P(-2, 3, 7), 19)$, $(P(-2, 3, 7), 18)$, and $(P(-2, 3, 7), 17)$, respectively.

Proof of Proposition 3.5. (1) We see from Figure 3.6 that $(n-1)$ -twist along c^{-1} converts $T_{-3,2}$ to the twist knot $Tw(n)$, i.e. $K_{n-1}^{-1} = Tw(n)$. Since $\text{lk}(c^{-1}, T_{-3,2}) = 0$, the surgery slopes do not change under the twisting.

FIGURE 3.6. $T_{-3,2} \cup c^{-1}$

(2) The sequence of isotopies in Figures 3.7, 3.8, and 3.9 shows that K_1^{-6} is the $(-2, 3, 7)$ pretzel knot, i.e. 1-twist along c^{-6} converts $T_{-3,2}$ to $P(-2, 3, 7)$. The surgery slopes 17, 18, 19 are obtained from Proposition 3.4 by setting $m = -6$, $n = 1$. \square (Proposition 3.5)

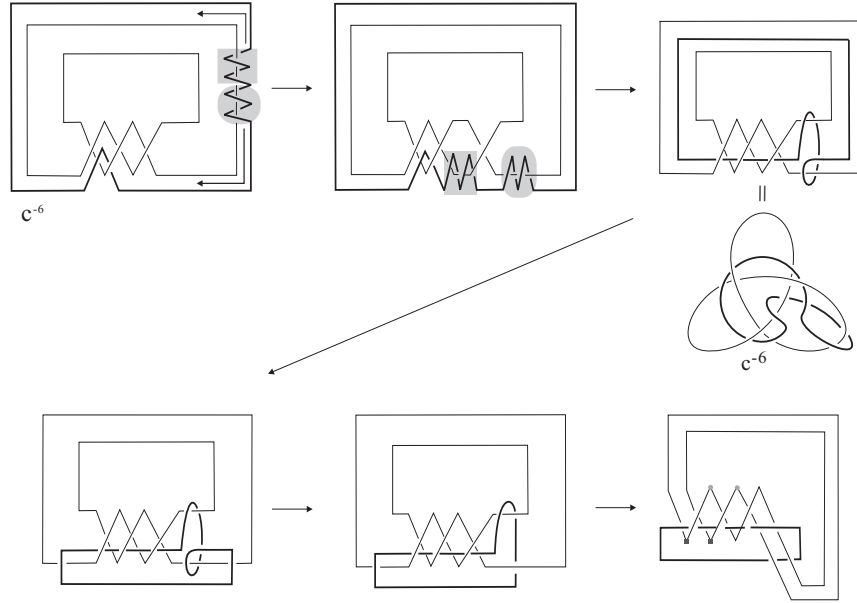
FIGURE 3.7. Isotopy of $T_{-3,2} \cup c^{-6}$

Figure 3.10 illustrates the subnetwork generated by twists along the seiferters c^m, c^{m+1}, c^{m+2} for $(T_{-3,2}, m)$.

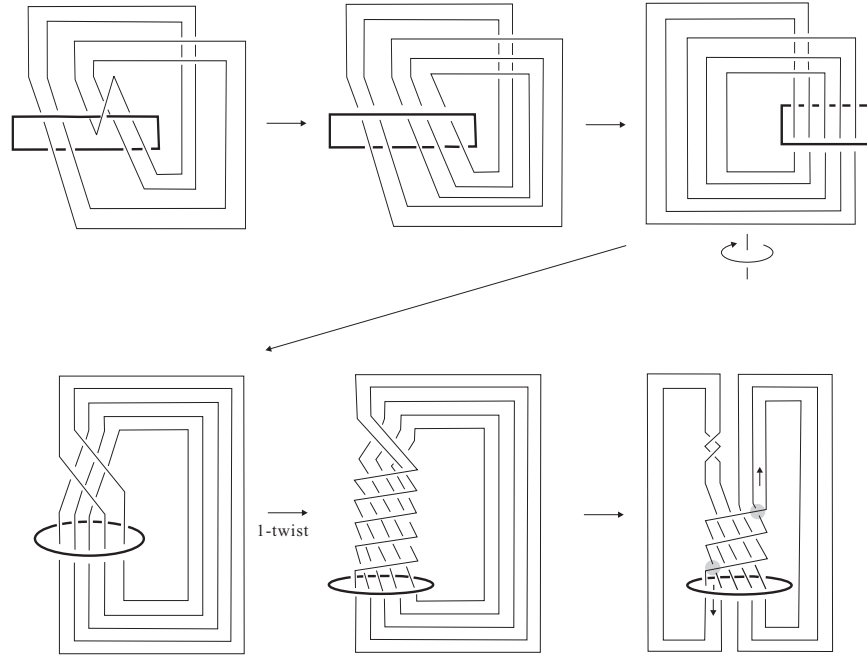
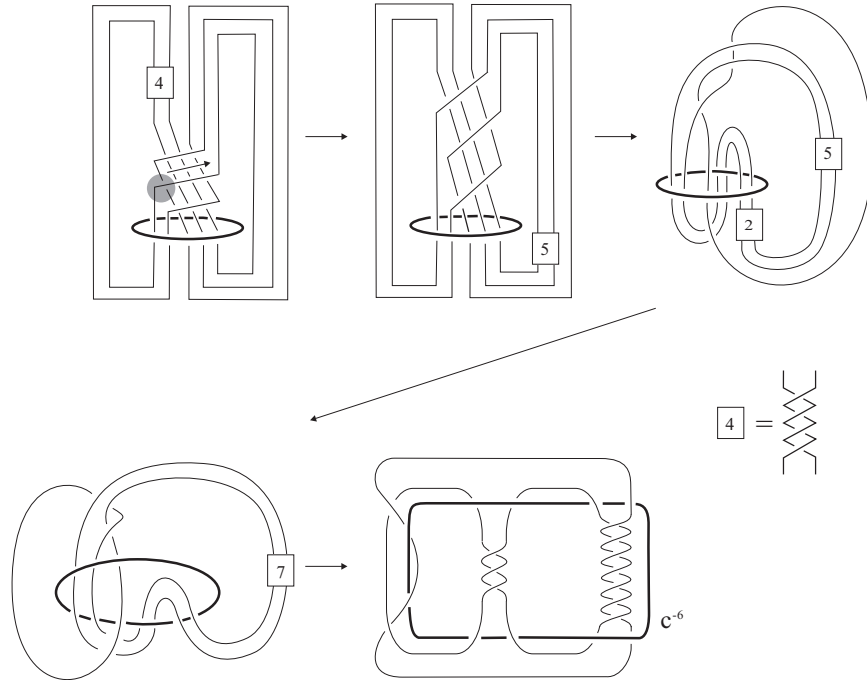


FIGURE 3.8. Continued from Figure 3.7

FIGURE 3.9. Continued from Figure 3.8: 1-twist of $T_{-3,2}$ along c^{-6}

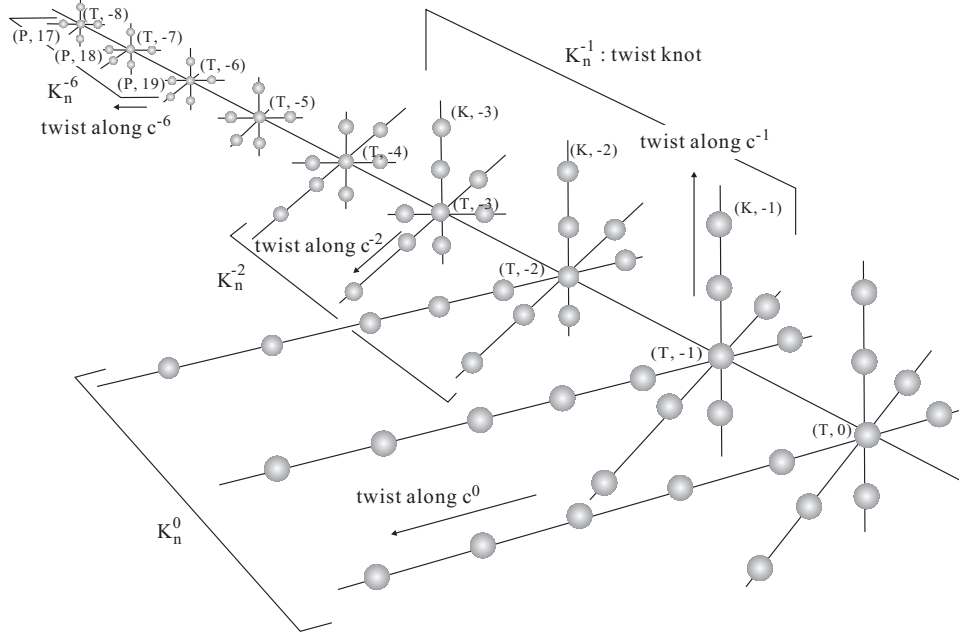


FIGURE 3.10. Subnetwork generated by c^m, c^{m+1}, c^{m+2} , where $T = T_{-3,2}$, $P = P(-2, 3, 7)$, and K is the figure-eight knot.

We calculate the indices of the exceptional fibers of $K_n^m(m+1-i+n(m+1)^2)$ where $i = 1, 2, 3$, and obtain Proposition 3.6 below. This proposition will be used to determine when K_n^m is hyperbolic.

Proposition 3.6. *For any integers m, n the following hold.*

- (1) $K_n^m(m+n(m+1)^2)$ is a Seifert fiber space over the base orbifold $S^2(2, 3, |n(m+2)(m+6)+m+n+6|)$.
- (2) $K_n^m(m-1+n(m+1)^2)$ is a Seifert fiber space over the base orbifold $S^2(2, |m+5|, |3n(m+3)-2n+3|)$.
- (3) $K_n^m(m-2+n(m+1)^2)$ is a Seifert fiber space over the base orbifold $S^2(3, |m+4|, |2n(m+4)-3n+2|)$.

Proof of Proposition 3.6. The Seifert fiber spaces $K_n^m(m+1-i+n(m+1)^2)$ ($i = 1, 2, 3$) are obtained from $T_{-3,2}(m+1-i)$ by n -twist along c^m . The seiferter $c^m = c_i^{m+1-i}$ is isotopic in $T_{-3,2}(m+1-i)$ to c_μ , s_{-3} , or s_2 according as $i = 1, 2$, or 3 . Recall that $T_{-3,2}(m+1-i)$ has a Seifert fibration over $S^2(2, 3, |m+7-i|)$ in which c_μ , s_{-3} , and s_2 are fibers of indices $|m+7-i|$, 3 , and 2 , respectively.

(1) We let $i = 1$ in the paragraph above. Then $K_n^m(m+n(m+1)^2)$ is obtained from $T_{-3,2}(m)$ by a surgery along the exceptional fiber of index $|m+6|$, and thus has a Seifert fibration over $S^2(2, 3, x)$ for some x . In $T_{-3,2}(m)$, c_1^m is isotopic to c_μ and further to the core of the filled solid torus U . We set $f : H_1(\partial N(c_1^m)) \rightarrow H_1(\partial U) = H_1(\partial N(T_{-3,2}))$ to be the homomorphism induced by this isotopy. Let (μ_c, λ_c) , (μ, λ) , and (μ', λ') be preferred meridian-longitude pairs

of $N(c_1^m)$, $N(c_\mu)$, and $N(T_{-3,2})$, respectively. Since c_1^m is obtained from c_μ by an m -move, Proposition 2.2(3) shows that 0-framing of c_μ becomes $(2\text{lk}(T_{-3,2}, c_\mu) + m)$ -framing of c_1^m after isotopy; here, $\text{lk}(T_{-3,2}, c_\mu) = 1$ where $T_{-3,2}$ and c_μ are oriented so as to satisfy the assumption in Proposition 2.2(3). Hence, the isotopy moving $N(c_1^m)$ to $N(c_\mu)$ sends μ_c, λ_c to $\mu, \lambda - (m+2)\mu$ curves on $\partial N(c_\mu)$. There is an annulus in $S^3 - \text{int}N(T_{-3,2} \cup c_\mu)$ connecting $\lambda (\subset \partial N(c_\mu))$ and $\mu' (\subset \partial N(T_{-3,2}))$. Since μ' is a longitude of U , the annulus extends to an annulus A connecting c_μ and the core of U . Isotope $N(c_\mu)$ to U along A . Then, μ is sent to $\lambda' + m\mu'$ curve (a meridian of U), and λ is sent to $-\mu'$ curve; in fact, $[\lambda' + m\mu'] \cdot [-\mu'] = -[\lambda'] \cdot [\mu'] = [\mu'] \cdot [\lambda'] = 1$. Combining these, we obtain $f([\mu_c]) = [\lambda'] + m[\mu']$ and $f([\lambda_c]) = -[\mu'] - (m+2)[\lambda' + m\mu']$. Hence, the image of the $(-\frac{1}{n})$ -surgery slope on $\partial N(c_1^m)$ is $f([-n\lambda_c + \mu_c]) = n([\mu'] + (m+2)[\lambda' + m\mu']) + [\lambda' + m\mu'] = (n(m+2)+1)[\lambda'] + (mn(m+2)+m+n)[\mu']$. On the other hand, a regular fiber on $\partial N(T_{-3,2})$ represents $[\lambda'] - 6[\mu']$. Then, the index x is $|f([\lambda_c - n\mu_c]) \cdot ([\lambda'] - 6[\mu'])|$. Computing this gives the claimed result.

(2) $K_n^m(m-1+n(m+1)^2)$ is obtained from $T_{-3,2}(m-1)$ by a surgery along the exceptional fiber s_{-3} of index 3, and thus has a Seifert fibration over $S^2(2, |m+5|, y)$ for some y . In $T_{-3,2}(m-1)$, c_2^{m-1} is isotopic to s_{-3} . We set $f : H_1(\partial N(c_2^{m-1})) \rightarrow H_1(\partial N(s_{-3}))$ to be the homomorphism induced by this isotopy. Let (μ_c, λ_c) , (μ, λ) be preferred meridian-longitude pairs of $N(c_2^{m-1})$, $N(s_{-3})$, respectively. Since c_2^{m-1} is obtained from s_{-3} by an $(m-1)$ -move, by Proposition 2.2(3) 0-framing of s_{-3} becomes $(2\text{lk}(T_{-3,2}, s_{-3}) + m - 1)$ -framing of c_2^{m-1} after isotopy; here, $\text{lk}(T_{-3,2}, c_2^{m-1}) = 2$. The reverse isotopy gives $f([\mu_c]) = [\mu]$ and $f([\lambda_c]) = [\lambda] - (m+3)[\mu]$. The image of the $(-\frac{1}{n})$ -surgery slope on $\partial N(c_2^{m-1})$ is $f([-n\lambda_c + \mu_c]) = -n[\lambda] + (n(m+3)+1)[\mu]$. On the other hand, a regular fiber on $\partial N(s_{-3})$ represents $-3[\lambda] + 2[\mu]$. Then, the index y is $|f([-n\lambda_c + \mu_c]) \cdot (-3[\lambda] + 2[\mu])|$. Assertion (2) follows from computation.

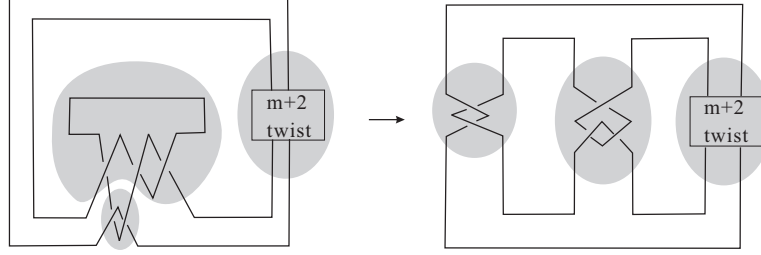
(3) The proof proceeds in the same manner as in (2) by replacing $m-1$ with $m-2$, c_2^{m-1} with c_3^{m-2} , and s_{-3} with s_2 . Note that $K_n^m(m-2+n(m+1)^2)$ is obtained from $T_{-3,2}(m-2)$ by a surgery along the exceptional fiber s_2 of index 2, and thus has a Seifert fibration over $S^2(3, |m+4|, z)$ for some z . In $T_{-3,2}(m-2)$, c_3^{m-2} is isotopic to s_2 . Let $f : H_1(\partial N(c_3^{m-2})) \rightarrow H_1(\partial N(s_2))$ be the homomorphism induced by the isotopy. Since $\text{lk}(T_{-3,2}, s_2) = 3$, Proposition 2.2(3) implies that $f([\mu_c]) = [\mu]$ and $f([\lambda_c]) = [\lambda] - (m+4)[\mu]$. The image of the $(-\frac{1}{n})$ -surgery slope on $\partial N(c_3^{m-2})$ is $f([-n\lambda_c + \mu_c])$. A regular fiber on $\partial N(s_2)$ represents $2[\lambda] - 3[\mu]$. Then, $z = |f([-n\lambda_c + \mu_c]) \cdot (2[\lambda] - 3[\mu])|$. Computation gives the claimed result.

□(Proposition 3.6)

We determine when c^m is a hyperbolic seiferter and K_n^m is a hyperbolic knot.

Proposition 3.7. *The link $T_{-3,2} \cup c^m$ is a hyperbolic link in S^3 if and only if $m \notin \{-5, -4, -3, -2\}$. In fact, c^{-4} , c^{-3} , c^{-2} are the same as the basic seiferters s_2 , s_{-3} , c_μ , respectively; c^{-5} is the $(-1, 2)$ cable of s_{-3} . In [5, Corollary 3.15(2)], a $(1, \frac{p \pm \varepsilon}{2})$ cable of the basic seiferter s_p for $T_{p,2}$ is called $s_{p,\varepsilon}$, where $\varepsilon = \pm 1$. Thus c^{-5} is $s_{-3,-1}$.*

Proof of Proposition 3.7. We first observe that $T_{-3,2} \cup c^m = T_{-3,2} \cup c_1^m$ in the last picture of Figure 3.2 is isotopic to the Montesinos link $M(-\frac{1}{2}, \frac{2}{3}, \frac{1}{2m+4})$ given in Figure 3.11 below.

FIGURE 3.11. $T_{-3,2} \cup c^m$ is isotopic to $M(-\frac{1}{2}, \frac{2}{3}, \frac{1}{2m+4})$.

If $m = -2$, then c^{-2} is a meridian of $T_{-3,2}$ and the exterior $S^3 - \text{int}N(T_{-3,2} \cup c^{-2})$ contains an essential torus. If $2m + 4 \neq 0$, then the Montesinos link $T_{-3,2} \cup c^m$ is formed by three rational tangles.

Assume that $T_{-3,2} \cup c^m$ is a Seifert link, i.e. the exterior admits a non-degenerate Seifert fibration. Then it turns out that c^m is a non-meridional basic seifert for $T_{-3,2}$. If $c^m = s_{-3}$, then $|\text{lk}(T_{-3,2}, c^m)| = |m + 1|$ equals to 2; if $c^m = s_2$, then $|m + 1| = 3$. In the former case, $m = 1, -3$ and in the latter $m = 2, -4$. In fact, c^{-3} is the basic seifert s_{-3} for $T_{-3,2}$. We show that c^1 is not a basic seifert. Note that $T_{-3,2} \cup c^1 = M(-\frac{1}{2}, \frac{2}{3}, \frac{1}{6})$ and $T_{-3,2} \cup c^{-3} = M(-\frac{1}{2}, \frac{2}{3}, -\frac{1}{2})$. The 2-fold branched covers of S^3 along these links have distinct base orbifolds $S^2(2, 3, 6)$ and $S^2(2, 3, 2)$. For a small Seifert fiber space its Seifert fibrations over S^2 are uniquely determined up to fiber preserving homeomorphism [12, 14]. Thus $T_{-3,2} \cup c^{-3}$ is not isotopic to $T_{-3,2} \cup c^1$ in S^3 . It follows that c^1 is not a basic seifert for $T_{-3,2}$. By the same argument we can check that c^{-4} is the basic seifert s_2 for $T_{-3,2}$, and c^2 is not a basic seifert for $T_{-3,2}$. Hence, $T_{-3,2} \cup c^m$ is not a Seifert link if $m \notin \{-4, -3\}$.

In the following, assume that $m \notin \{-2, -3, -4\}$. Then, if $T_{-3,2} \cup c^m = M(-\frac{1}{2}, \frac{2}{3}, \frac{1}{2m+4}) = M(\frac{1}{2}, -\frac{1}{3}, \frac{1}{2m+4})$ is not hyperbolic, [19, Corollary 5] shows that it is isotopic to $M(\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6})$ or its mirror image. Comparing the indices of the exceptional fibers in the 2-fold branched cover along $T_{-3,2} \cup c^m$ with that along $M(\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6})$, we see that $2m + 4 = \pm 6$ and so $m = 1, -5$. The 2-fold branched cover along $T_{-3,2} \cup c^1 = M(\frac{1}{2}, -\frac{1}{3}, \frac{1}{6})$ and that along $M(\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6})$ are not homeomorphic because the Euler numbers of these Seifert fibrations are $\frac{1}{2} - \frac{1}{3} + \frac{1}{6} = \frac{1}{3}$ and $\frac{1}{2} - \frac{1}{3} - \frac{1}{6} = 0$, i.e. distinct up to sign. This implies that $T_{-3,2} \cup c^m$ is hyperbolic if $m = 1$. It follows $m = -5$. Figure 1.4 in [19] shows that the exterior of $T_{-3,2} \cup c^{-5} = M(\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6})$ in S^3 contains an essential torus, and furthermore c^{-5} is the $(-1, 2)$ cable of s_{-3} for $T_{-3,2}$. \square (Proposition 3.7)

Corollary 3.8. (1) If $m \leq -8$ or $-1 \leq m$, then c^m , c^{m+1} , and c^{m+2} are distinct hyperbolic seiferters for $(T_{-3,2}, m)$.

(2) For any integer $m \neq -4$, $(T_{-3,2}, m)$ has a hyperbolic seifert.

Proof of Corollary 3.8. Proposition 3.7 gives Table 3.1 below. This table implies assertion (1). The table shows that $(T_{-3,2}, m)$ has a hyperbolic seifert if $m \notin \{-4, -5\}$. For $m = -5$, it is shown in [6] that the lens surgery $(T_{-3,2}, -5)$ has

a hyperbolic seifert.

□(Corollary 3.8)

TABLE 3.1. “h” means a hyperbolic seifert

seifert for $(T_{-3,2}, m)$	$m \leq -8$	-7	-6	-5	-4	-3	-2	$-1 \leq m$
$c^m = c_1^m$	h	h	h	$s_{-3,-1}$	s_2	s_{-3}	c_μ	h
$c^{m+1} = c_2^m$	h	h	$s_{-3,-1}$	s_2	s_{-3}	c_μ	h	h
$c^{m+2} = c_3^m$	h	$s_{-3,-1}$	s_2	s_{-3}	c_μ	h	h	h

We do not know whether $(T_{-3,2}, -4)$ has a hyperbolic seifert or not. In Section 4, we show that it has at least six hyperbolic annular pairs of seiferters.

Proposition 3.9. *The knot K_n^m is a hyperbolic knot in S^3 if and only if $m \notin \{-5, -4, -3, -2\}$, $n \neq 0$, and $(m, n) \neq (-1, -1)$.*

Proof of Proposition 3.9. (1) If $m = -5, -4, -3$, or -2 , then the torus decomposition pieces of $X = S^3 - N(T_{-3,2} \cup c^m)$ are Seifert fiber spaces. It follows that the exterior of K_n^m , the result of a Dehn filling of X , is not hyperbolic for any m, n . If $n = 0$, K_0^m is $T_{-3,2}$. If $m = n = -1$, then K_{-1}^{-1} is the twist knot $Tw(0)$, a trivial knot (see Figure 3.5). The “only if” part of Proposition 3.9 is proved.

(2) To prove the “if” part we start with a proof of Lemma 3.10 below.

Lemma 3.10. *No satellite knot has three successive Seifert fibered surgeries.*

Proof of Lemma 3.10. This follows from [16, 17]. By [17, Theorem 1.2] a satellite knot which is not cabled exactly once admits at most two (integral) Seifert fibered surgeries and the slopes are successive. Assume that K is an (r, s) cable knot and m -surgery on K is a Seifert fibered surgery. Then, if $K(m)$ contains an essential torus, by [16, Theorem 1.2] K is the $(2pq \pm 1, 2)$ cable of a torus knot $T_{p,q}$ and $m = 4pq$. If $K(m)$ contains no essential torus, then the proof of Theorem 1.4 in [16] shows that $m = rs \pm 1$. Therefore, if K is not a $(2pq \pm 1, 2)$ cable of $T_{p,q}$, it has exactly two Seifert fibered surgeries; if K is a $(2pq \pm 1, 2)$ cable of $T_{p,q}$, it has exactly three Seifert fibered surgeries $4pq, 4pq \pm 1, 4pq \pm 3$, which are not successive. (In the latter case, $K(4pq \pm 2)$ is a connected sum of two lens spaces.) □(Lemma 3.10)

Claim 3.11. *K_n^{-6} is a hyperbolic knot for $n \neq 0$.*

Proof of Claim 3.11. If $m = -6$, then by Proposition 3.6 $K_n^{-6}(25n - i)$ has a Seifert fibration over $S^2(2, 3, |n|)$, $S^2(2, 1, |11n - 3|)$, or $S^2(3, 2, |7n - 2|)$ according as $i = 6, 7$, or 8 . All indices of the exceptional fibers of these fibrations are nonzero, so that $(K_n^{-6}, 25n - i)$ ($i = 6, 7, 8$) are three successive Seifert fibered surgeries for $n \neq 0$. Hence, K_n^{-6} ($n \neq 0$) is not a satellite knot by Lemma 3.10. We also see that K_n^{-6} is a nontrivial knot because $K_n^{-6}(25n - 8)$ is not a lens space.

For a nontrivial torus knot $T_{p,q}$, $T_{p,q}(r)$ ($r \in \mathbb{Z}$) is a lens space if and only if $r = pq \pm 1$; $T_{p,q}(pq)$ is a connected sum of two lens spaces. Now $K_n^{-6}(25n - 7)$ is a lens space, so that $K_n^{-6}(25n - 8)$ or $K_n^{-6}(25n - 6)$ is a connected sum of two lens spaces. By the assumption $n \neq 0$, this is impossible. Hence K_n^{-6} is not a torus

knot, and thus a hyperbolic knot for $n \neq 0$.

□(Claim 3.11)

We thus assume that $m \notin \{-6, -5, -4, -3, -2\}$, $n \neq 0$, and $(m, n) \neq (-1, -1)$.

Claim 3.12. $(K_n^m, m+1-i+n(m+1)^2)$ ($i = 1, 2, 3$) are three successive small Seifert fibered surgeries.

Then, Lemma 3.10 shows that K_n^m is not a satellite knot or a trivial knot.

Proof of Claim 3.12. Proposition 3.6 shows that the base orbifolds of $K_n^m(m+1-i+n(m+1)^2)$ is $S^2(2, 3, a)$, $S^2(2, |m+5|, b)$, or $S^2(3, |m+4|, c)$ where $a = |n(m+2)(m+6) + m + n + 6|$, $b = |3n(m+3) - 2n + 3|$, $c = |2n(m+4) - 3n + 2|$, according as $i = 1, 2$, or 3 . Since $|m+5| \geq 2$ and $|m+4| \geq 2$, it is sufficient to show that a, b, c are greater than or equal to 2.

Let $f(x) = nx^2 + (8n+1)x + 13n+6$ ($x \in \mathbb{R}$); then $|f(m)| = a$. If $n > 0$, then the axis $y = -\frac{8n+1}{2n}$ of the parabola $y = f(x)$ lies between -5 and -4 . Hence, $f(m) \geq f(-7) = 6n - 1 \geq 5$, where $m \leq -7, m \geq -1$. If $n < 0$, then the fact that $-4 < -\frac{8n+1}{2n} < -3$ implies that $f(m) \leq f(-1) = 6n + 5 \leq -1$; the last equality holds only if $n = -1$. Since $(m, n) \neq (-1, -1)$, these results imply $a = |f(m)| \geq 2$.

Let $g(x) = 3nx + 7n + 3$ ($x \in \mathbb{R}$); then $b = |g(m)|$ and the zero point $x = -\frac{7n+3}{3n}$ of the linear function $g(x)$ lies between -4 and -1 . We see that $|g(m)| \geq |g(-1)| = |4n + 3| \geq 1$, where $m \leq -7, m \geq -1$. If $|g(m)| = 1$, then $(m, n) = (-1, -1)$, a contradiction. Thus $b \geq 2$.

Regarding $c = h(x)$ where $h(x) = 2nx + 5n + 2$, the zero point $x = -\frac{5n+2}{2n}$ of h lies between -4 and -1 . Thus, for $m \leq -7$ and $m \geq -1$ we have $|h(m)| \geq |h(-1)| = |3n + 2| \geq 1$. The equality $|h(m)| = 1$ holds only if $(m, n) = (-1, -1)$, an excluded case. Hence, $c \geq 2$ as desired. □(Claim 3.12)

Assume for a contradiction that K_n^m is a torus knot $T_{p,q}$ where $|p| > q \geq 2$ for some m, n satisfying our assumption. For simplicity, set $d = m+1+n(m+1)^2$. Then $K_n^m(d-i)$ admits a Seifert fibration over $S^2(|p|, q, |pq-d+i|)$ for $i \in \{1, 2, 3\}$. Thus the unordered triples of the indices of exceptional fibers satisfy $(2, 3, a) = (|p|, q, |pq-d+1|)$, $(2, |m+5|, b) = (|p|, q, |pq-d+2|)$, and $(3, |m+4|, c) = (|p|, q, |pq-d+3|)$ for some integers m, n , $|p| > q \geq 2$. Since $|pq-d+i| \neq 0, 1$ for $i = 1, 2, 3$ (Claim 3.12), the indices $|pq-d+1|, |pq-d+2|, |pq-d+3|$ are mutually distinct. Hence, all triples have 2 and 3 in common, so that $|m+4| = 2$ or $c = 2$. The former case implies $m = -2, -6$, a contradiction. The latter case $c = |2nm + 5n + 2| = 2$ implies $(m, n) = (-3, 4), (-2, -4)$, a contradiction. Therefore, K_n^m is neither a satellite knot nor a torus knot, so that K_n^m is a hyperbolic knot. □(Proposition 3.9)

Proposition 3.9 and its proof imply the following theorem, which generalizes a previous result in [18].

Theorem 3.13. *The following (1) and (2) hold.*

- (1) *For any integer m , there is a hyperbolic knot K such that (K, m) , $(K, m+1)$, $(K, m+2)$ are small Seifert fibered surgeries.*
- (2) *If $m \neq -2$, the hyperbolic knot K in (1) above can be chosen so that three successive surgeries in (1) arise from three successive Seifert surgeries on $T_{-3,2}$ by twisting along a common seiferters.*

Proof of Theorem 3.13. Claim 3.12 shows that (K_n^0, n) , $(K_n^0, n-1)$, $(K_n^0, n-2)$ are small Seifert fibered surgeries, where $n \neq 0$. By Proposition 3.9 K_n^0 ($n \neq 0$) is hyperbolic. Regarding $0-$, $(-1)-$, $(-2)-$ surgeries, take the mirror images of $(K_2^0, 2)$, $(K_2^0, 1)$, $(K_2^0, 0)$. Then we obtain $(-2)-$, $(-1)-$, 0 -surgeries on the mirror image K' of K_2^0 , small Seifert fibered surgeries on a hyperbolic knot. Note that these surgeries on K' also arise from three successive surgeries on $T_{-3,2}$ after twisting along a common seifert. \square (Theorem 3.13)

Question 3.14. Is the assumption $m \neq -2$ in Theorem 3.13(2) necessary?

4. ANNULAR PAIRS FOR SEIFERT SURGERIES ON A TREFOIL KNOT

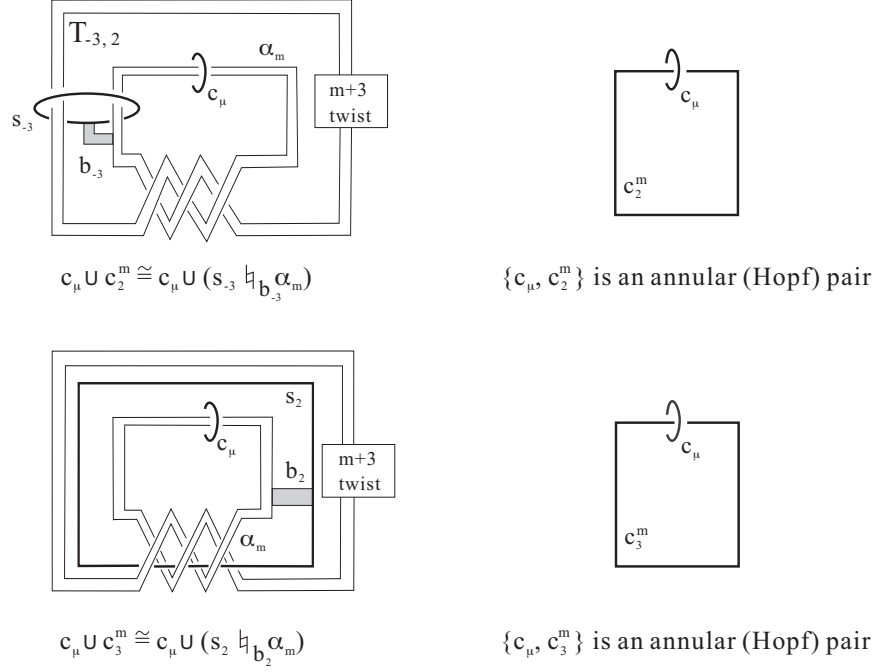
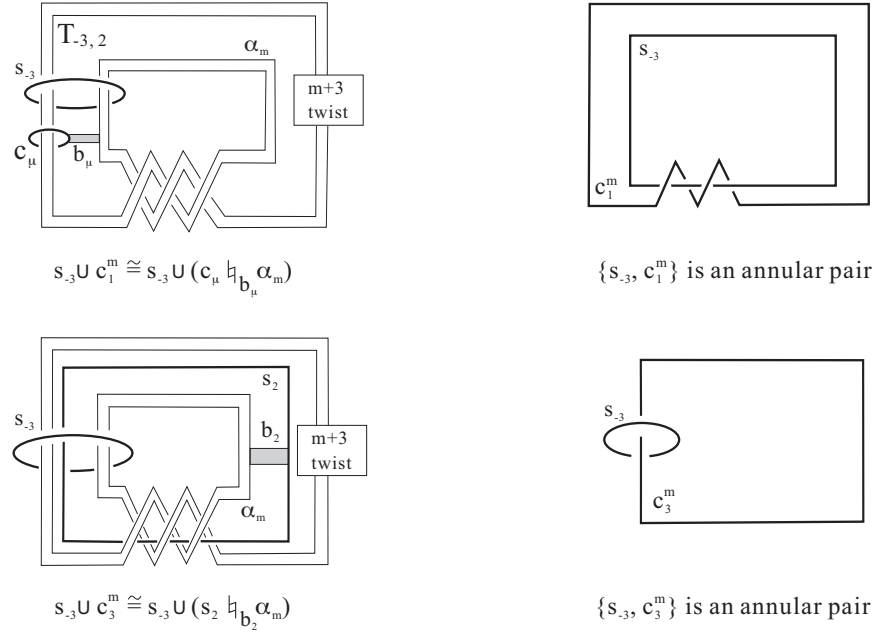
As observed in Section 3, $(T_{-3,2}, m)$ has six seiferters $c_\mu, s_{-3}, s_2, c_1^m, c_2^m, c_3^m$. We completely determined which are basic seiferters or hyperbolic seiferters (Proposition 3.7). In this section, we obtain annular pairs of seiferters by placing any two of these six seiferters in adequate positions. We first prove Lemma 4.1 below by applying m -moves to basic annular pairs for $(T_{-3,2}, m)$.

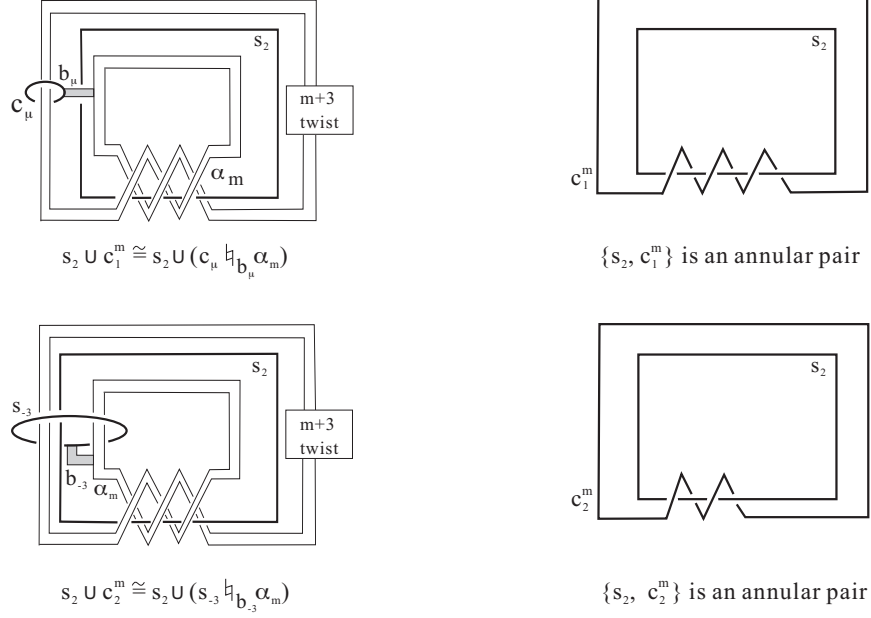
Lemma 4.1. $\{c_\mu, c_2^m\}$, $\{c_\mu, c_3^m\}$ in Figure 4.1, $\{s_{-3}, c_1^m\}$, $\{s_{-3}, c_3^m\} (\neq \{s_{-3}, c_3^{-5}\})$ in Figure 4.2, $\{s_2, c_1^m\}$, $\{s_2, c_2^m\} (\neq \{s_2, c_2^{-5}\})$ in Figure 4.3, $\{c_1^m, c_2^m\}$, $\{c_1^m, c_3^m\}$ and $\{c_2^m, c_3^m\}$ in Figure 4.4 are annular pairs of seiferters for $(T_{-3,2}, m)$ for any m . Each of these pairs is isotopic in $T_{-3,2}(m)$ to a basic annular pair for $(T_{-3,2}, m)$.

Remark 4.2. The excluded pairs $\{s_{-3}, c_3^{-5}\}$ and $\{s_2, c_2^{-5}\}$ in Figures 4.3, 4.4 with $m = -5$ cobound annuli in $S^3 - \text{int}N(T_{-3,2})$. By Remark 1.1 they are not regarded as annular pairs of seiferters for $(T_{-3,2}, -5)$.

Proof of Lemma 4.1. In Figures 4.1–4.3, each pair uses one band, and is obtained from a basic annular pair of seiferters by an m -move. Then Proposition 2.4(2) shows that such a pair is a pair of seiferters. Regarding pairs in Figure 4.4, each of them uses two mutually disjoint bands and satisfies the assumption in Corollary 2.5. It then follows from Corollary 2.5 that each pair in Figure 4.4 is a pair of seiferters.

The fact that all the pairs cobound annuli in S^3 is shown in Figures 4.1–4.4. It remains to show that all the pairs in Lemma 4.1 do not cobound annuli in $S^3 - \text{int}N(T_{-3,2})$. Recall that $|\text{lk}(T_{-3,2}, c_\mu)| = 1$, $|\text{lk}(T_{-3,2}, s_{-3})| = 2$, $|\text{lk}(T_{-3,2}, s_2)| = 3$, and $|\text{lk}(T_{-3,2}, c_i^m)| = |m+i|$. Hence, if a pair of seiferters in Figures 4.1–4.3 satisfies the condition that its components have the same linking numbers with $T_{-3,2}$ (up to sign), then the pair is one of the following list: $\{c_\mu, c_2^m\}$ with $m = -3, -1$, $\{c_\mu, c_3^m\}$ with $m = -4, -2$, $\{s_{-3}, c_1^m\}$ with $m = -3, 1$, $\{s_{-3}, c_3^m\}$ with $m = -5, -1$, $\{s_2, c_1^m\}$ with $m = -4, 2$, $\{s_2, c_2^m\}$ with $m = -5, 1$. Then, Proposition 2.6(1) guarantees that the pairs not on the above list are (relevant) annular pairs of seiferters for $(T_{-3,2}, m)$. Proposition 2.6(2) shows that if $T_{-3,2}(m)$ is not a lens space and a pair of seiferters $\{c_1, c_2\}$ for $(T_{-3,2}, m)$ cobounds an annulus in $S^3 - T_{-3,2}$, then c_1 and c_2 are regular fibers in $T_{-3,2}(m)$. Each seifert in a pair listed above is an exceptional fiber in $T_{-3,2}(m)$ if $T_{-3,2}(m)$ is not a lens space. This is because each basic seifert is a (possibly degenerate) exceptional fiber in $T_{-3,2}(m)$ if $T_{-3,2}(m)$ is not a lens space. Hence, Proposition 2.6(2) narrows the list above to the cases when $T_{-3,2}(m)$ is a lens space: $\{s_{-3}, c_3^{-5}\}$ and $\{s_2, c_2^{-5}\}$; both are pairs of seiferters for $(T_{-3,2}, -5)$. \square (Lemma 4.1)

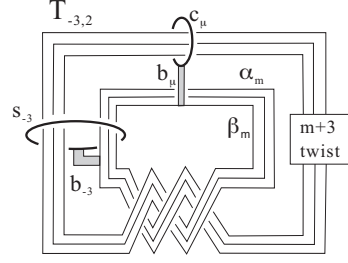
FIGURE 4.1. Annular pairs of seiferters $\{c_\mu, c_2^m\}$, $\{c_\mu, c_3^m\}$ for $(T_{-3,2}, m)$ FIGURE 4.2. Annular pairs of seiferters $\{s_{-3}, c_1^m\}$, $\{s_{-3}, c_3^m\}$ for $(T_{-3,2}, m)$

FIGURE 4.3. Annular pairs of seiferters $\{s_2, c_1^m\}$, $\{s_2, c_2^m\}$ for $(T_{-3,2}, m)$

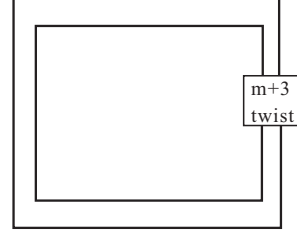
Assume $|m + 6| > 3$. Then, no matter where the two seiferters c_μ and c_1^m are placed in $S^3 - T_{-3,2}$, $\{c_\mu, c_1^m\}$ cannot be a pair of seiferters for $(T_{-3,2}, m)$ for the following reason. First note that any Seifert fibration on $T_{-3,2}(m)$ has three exceptional fibers with mutually distinct indices $2, 3, |m + 6|$. Since c_μ and c_1^m are isotopic in $T_{-3,2}(m)$, their exteriors in $T_{-3,2}(m)$ are homeomorphic. Therefore, if $T_{-2,3}(m)$ has a Seifert fibration in which c_μ and c_1^m are fibers simultaneously, they are regular fibers. Then $T_{-2,3}(m) - \text{int}N(c_\mu)$ is a Seifert fiber space over $D^2(2, 3, |m + 6|)$. This contradicts the fact that $T_{-2,3}(m) - \text{int}N(c_\mu)$ is a Seifert fiber space over $D^2(2, 3)$. For the same reason, $\{s_{-3}, c_2^m\}$ and $\{s_2, c_3^m\}$ cannot be pairs of seiferters for $(T_{-3,2}, m)$, where $|m + 6| > 3$. On the other hand, for $m = -5, -7$, using the flexibility of Seifert fibrations on the lens space $T_{-3,2}(m)$, we obtain the following.

Lemma 4.3. *For $m = -5, -7$, assertions (1), (2), (3) below hold. Each (annular) pair of seiferters obtained below consists of fibers in a non-degenerate Seifert fibration of $T_{-3,2}(m)$.*

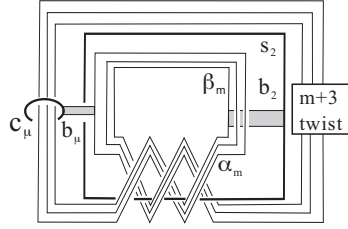
- (1) $\{c_\mu, c_1^{-5}\}$ in Figure 4.6 is an annular pair of seiferters for $(T_{-3,2}, -5)$; $\{c_\mu, c_1^{-7}\}$ in Figure 4.6 is a pair of seiferters, but not an annular pair of seiferters for $(T_{-3,2}, -7)$.
- (2) $\{s_{-3}, c_2^m\}$ in Figure 4.7 is a pair of seiferters for $(T_{-3,2}, m)$ for any integer p ; $\{s_{-3}, c_2^m\}$ is an annular pair of seiferters for $(T_{-3,2}, m)$ exactly when $p = 0, -1$.
- (3) $\{s_2, c_3^m\}$ in Figure 4.8 is a pair of seiferters for $(T_{-3,2}, m)$ for any integer p ; $\{s_2, c_3^m\}$ is an annular pair of seiferters for $(T_{-3,2}, m)$ exactly when $p = -1, -2$.



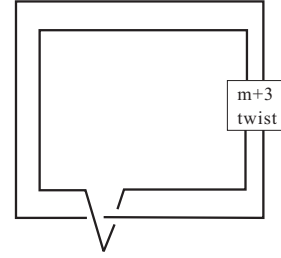
$$c_1^m \cup c_2^m \cong (c_\mu \natural_{b_\mu} \beta_m) \cup (s_{-3} \natural_{b_{-3}} \alpha_m)$$



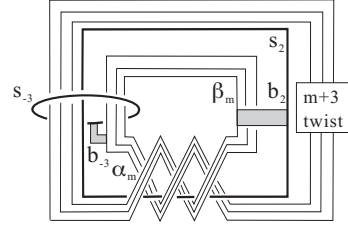
$\{c_1^m, c_2^m\}$ is an annular pair



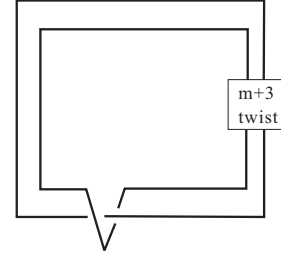
$$c_1^m \cup c_3^m \cong (c_\mu \natural_{b_\mu} \alpha_m) \cup (s_2 \natural_{b_2} \beta_m)$$



$\{c_1^m, c_3^m\}$ is an annular pair



$$c_2^m \cup c_3^m \cong (s_{-3} \natural_{b_{-3}} \alpha_m) \cup (s_2 \natural_{b_2} \beta_m)$$



$\{c_2^m, c_3^m\}$ is an annular pair

FIGURE 4.4. Annular pairs of seiferters $\{c_1^m, c_2^m\}$, $\{c_1^m, c_3^m\}$, $\{c_2^m, c_3^m\}$ for $(T_{-3,2}, m)$

Proof of Lemma 4.3. Take $m \in \{-5, -7\}$.

(1) The link $c_\mu \cup c_1^m$ in Figure 4.6 is isotopic in $T_{-3,2}(m)$ to the union of c_μ and the $(m+6, 1)$ cable of $N(c_\mu)$. Note that since $m+6 = \pm 1$, the $(m+6, 1)$ cable is the $(1, m+6)$ cable. For any integer n , an isotopy in $T_{-3,2}(n)$ sending c_μ to the core of the filled solid torus sends the $(1, n+6)$ cable of c_μ to the $(-6, 1)$ cable of $T_{-3,2}$; refer to [5, Corollary 3.15(7)] and its proof. Thus, $c_\mu \cup c_1^m$ is isotopic in $T_{-3,2}(m)$ to the union of the core of the filled solid torus and a regular fiber

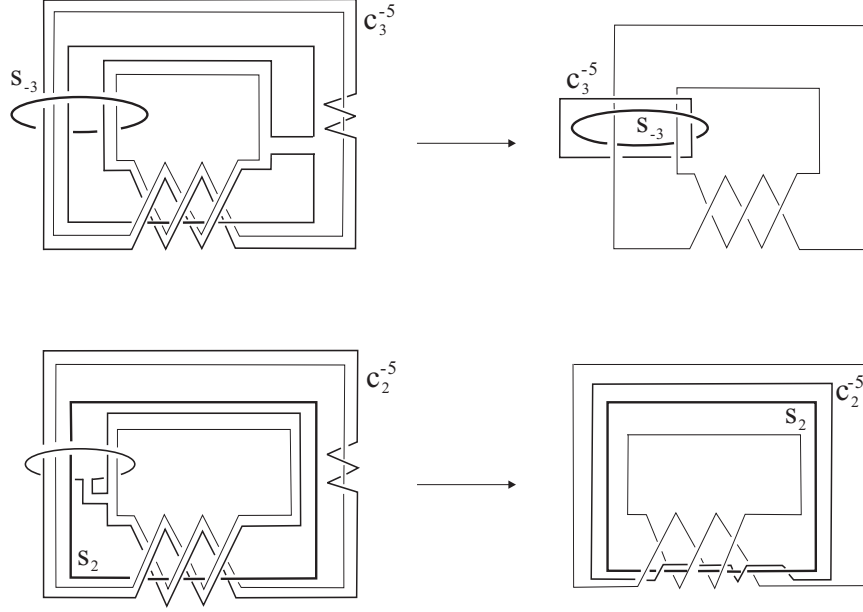


FIGURE 4.5. $\{s_{-3}, c_3^{-5}\}$ and $\{s_2, c_2^{-5}\}$ are irrelevant annular pairs of seiferters for $(T_{-3,2}, -5)$.

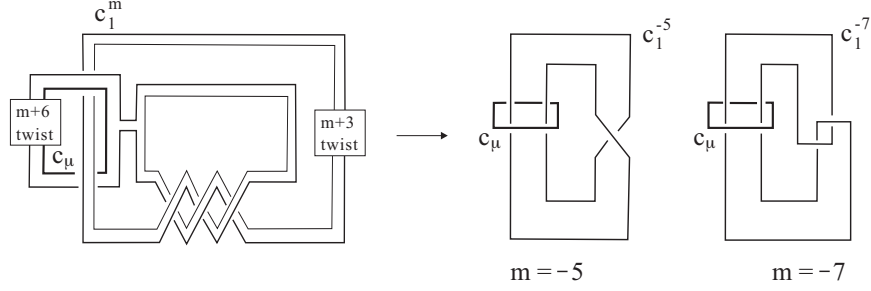


FIGURE 4.6. Pairs of seiferters $\{c_\mu, c_1^m\}$ for $(T_{-3,2}, m)$, where $m \in \{-5, -7\}$.

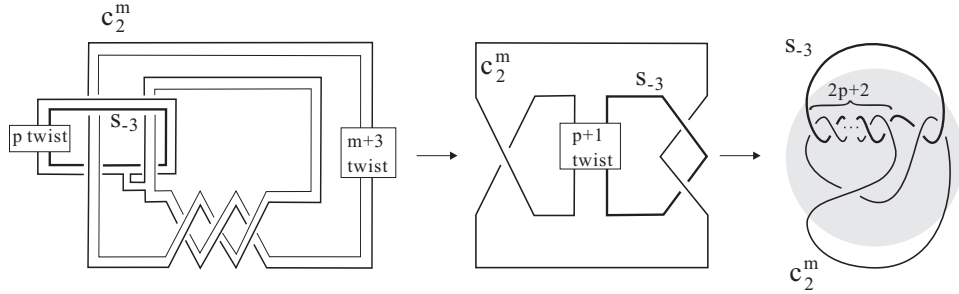
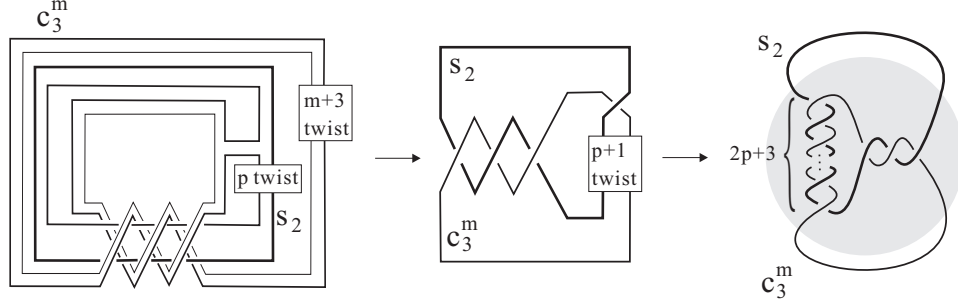


FIGURE 4.7. Pairs of seiferters $\{s_{-3}, c_2^m\}$ for $(T_{-3,2}, m)$, where $m \in \{-5, -7\}$.

FIGURE 4.8. Pairs of seiferters $\{s_2, c_3^m\}$ for $(T_{-3,2}, m)$, where $m \in \{-5, -7\}$.

in $S^3 - \text{int}N(T_{-3,2})$. Thus, the lens space $T_{-3,2}(m)$ has a non-degenerate Seifert fibration with c_μ and c_1^m regular fibers. As shown in Figure 4.6, $c_\mu \cup c_1^{-5}$ is the $(2, -4)$ torus link and cobounds an annulus. Note that such an annulus cannot be disjoint from $T_{-3,2}$ because $|\text{lk}(c_\mu, T_{-3,2})| = 1 \neq |m+1| = |\text{lk}(c_1^m, T_{-3,2})|$ for $m \in \{-5, -7\}$ by Proposition 2.6(1). Thus $\{c_\mu, c_1^{-5}\}$ is an annular pair. On the other hand, since $c_\mu \cup c_1^{-7}$ is the Whitehead link, $\{c_\mu, c_1^{-7}\}$ is not an annular pair.

(2) The lens space $T_{-3,2}(m)$ has a Seifert fibration with s_{-3} an exceptional fiber, so that $V' = T_{-3,2}(m) - \text{int}N(s_{-3})$ is a solid torus and obtained from the solid torus $V = S^3 - \text{int}N(s_{-3})$ by m -surgery on $T_{-3,2}$. Since $T_{-3,2}$ is the $(-3, 2)$ cable of V , a meridian of V' represents $[\mu] \pm 2(2[\lambda] - 3[\mu]) = \pm(4[\lambda] + m[\mu]) \in H_1(\partial V)$, where (μ, λ) is a preferred meridian-longitude pair of $V(\subset S^3)$. Note that $s_{-3} \cup c_2^m$ in Figure 4.7 is isotopic in $T_{-3,2}(m)$ to the union of s_{-3} and the $(p, 1)$ cable of s_{-3} in S^3 . Since $(p, 1) \neq (4, m)$, $T_{-3,2}(m)$ has a non-degenerate Seifert fibration in which s_{-3} and c_2^m are fibers. As shown in Figure 4.7, $s_{-3} \cup c_2^m$ is the 2-bridge link associated to $\frac{6p+4}{2p+1}$. It is a torus link exactly when $p = 0, -1$. Hence s_{-3} and c_2^m cobound an annulus exactly when $p = 0, -1$. Since $|\text{lk}(s_{-3}, T_{-3,2})| = 2 \neq |m+2| = |\text{lk}(c_2^m, T_{-3,2})|$ for $m \in \{-5, -7\}$, the annulus must intersect $T_{-3,2}$ by Proposition 2.6. Hence, $\{s_{-3}, c_2^m\}$ is an annular pair if and only if $p = 0, -1$.

(3) As in (2), $V' = T_{-3,2}(m) - \text{int}N(s_2)$ is a solid torus and obtained from the solid torus $S^3 - \text{int}N(s_2)$ by m -surgery on $T_{-3,2}$. Since $T_{-3,2}$ is the $(-2, 3)$ cable of V , a meridian of V' is a $(9, m)$ cable of s_2 . Since $(p, 1) \neq (9, m)$, $T_{-3,2}(m)$ has a non-degenerate Seifert fibration with s_2 and c_3^m fibers. Figure 4.8 shows that $s_2 \cup c_3^m$ is the 2-bridge link associated to $\frac{6p+10}{2p+3}$. It is a torus link exactly when $p = -1, -2$. Since $|\text{lk}(s_2, T_{-3,2})| = 3 \neq |m+3| = |\text{lk}(c_3^m, T_{-3,2})|$ for $m \in \{-5, -7\}$, $\{s_2, c_3^m\}$ is an annular pair if and only if $p = -1, -2$. \square (Lemma 4.3)

Let us determine which annular pairs given in Lemmas 4.1 and 4.3 are basic annular pairs.

Proposition 4.4. *Let $\{\alpha_m, \beta_m\}$ be an annular pair of seiferters in Figures 4.1–4.4 and 4.6–4.8, and assume that $\{\alpha_m, \beta_m\}$ is a basic annular pair of seiferters for $T_{-3,2}$. Then $m = -3, -4, -5, -6$, and one of the following holds.*

- (1) $\{\alpha_m, \beta_m\} = \{c_1^{-3}, c_2^{-3}\}$ in Figure 4.4 if $m = -3$.
- (2) $\{\alpha_m, \beta_m\} = \{c_1^{-4}, c_2^{-4}\}$, $\{c_1^{-4}, c_3^{-4}\}$ or $\{c_2^{-4}, c_3^{-4}\}$ in Figure 4.4 if $m = -4$.

- (3) $\{\alpha_m, \beta_m\} = \{c_2^{-5}, c_3^{-5}\}$ in Figure 4.4, $\{s_{-3}, c_2^{-5}\}$ with $p = -1$ in Figure 4.7, or $\{s_2, c_3^{-5}\}$ with $p = -2$ in Figure 4.8 if $m = -5$.
 (4) $\{\alpha_m, \beta_m\} = \{s_{-3}, c_3^{-6}\}$ in Figure 4.2 if $m = -6$.

Conversely, the pairs in (1)–(4) are basic annular pairs of seiferters for $T_{-3,2}$.

Proof of Proposition 4.4.

Case 1. $\{\alpha_m, \beta_m\}$ is an annular pair of seiferters in Figures 4.1–4.4.

The linking numbers between any two of $T_{-3,2}$, α_m , and β_m are given in Table 4.1 below.

TABLE 4.1. Linking numbers of seiferters in Figures 4.1–4.4

lk	$T_{-3,2}$	c_μ	s_{-3}	s_2	c_1^m	c_2^m
c_3^m	$m+3$	1	1	*	$m+4$	$m+4$
c_2^m	$m+2$	1	*	2	$m+3$	
c_1^m	$m+1$	*	2	3		
s_2	3					
s_{-3}	2					
c_μ	1					

Since $\{\alpha_m, \beta_m\}$ is a basic annular pair of seiferters for $T_{-3,2}$, the triple $(|\text{lk}(T_{-3,2}, \alpha_m)|, |\text{lk}(T_{-3,2}, \beta_m)|, |\text{lk}(\alpha_m, \beta_m)|)$ is equal to $(1, 2, 0)$, $(1, 3, 0)$ or $(2, 3, 1)$ according as $(\alpha_m, \beta_m) = (c_\mu, s_{-3})$, (c_μ, s_2) or (s_{-3}, s_2) . Checking the triples of linking numbers for the nine possible pairs, we obtain all cases listed in Proposition 4.4(1)–(4) and one unexpected case $\{s_{-3}, c_3^0\}$. However, the latter is not a basic annular pair because $c_3^0 (= c^2)$ is a hyperbolic seiferters for $(T_{-3,2}, 0)$ by Proposition 3.7(1). Conversely, the pairs in Proposition 4.4(1)–(4) are basic annular pairs of seiferters, as shown in Figures 4.9 and 4.10.

Case 2. $\{\alpha_m, \beta_m\}$ is an annular pair of seiferters in Figures 4.6–4.8.

Checking whether c_1^m, c_2^m, c_3^m ($m = -5, -7$) are basic seiferters by Table 3.1, we see that $m = -5$, and $\{c_\mu, c_1^m\}$ in Figure 4.6 is not a basic annular pair of seiferters. We also see that $\{\alpha_m, \beta_m\}$ is either $\{s_{-3}, c_2^m\}$ in Figure 4.7 or $\{s_2, c_3^m\}$ in Figure 4.8, and in either case it is the basic annular pair $\{s_{-3}, s_2\}$. If $\{s_{-3}, c_2^{-5}\}$ in Figure 4.7 is $\{s_{-3}, s_2\}$, then $p \in \{0, -1\}$ by Lemma 4.3 and the 2-bridge link $s_{-3} \cup c_2^{-5}$ in the right-most figure of Figure 4.6 is a Hopf link. It follows $p = -1$. Similarly, if $\{s_2, c_3^{-5}\}$ in Figure 4.8 is $\{s_{-3}, s_2\}$, then we see $p = -2$ from the right-most figure of Figure 4.6. Proposition 4.4 follows from Lemma 4.5 below. \square (Proposition 4.4)

Lemma 4.5. (1) $\{s_{-3}, c_2^{-5}\}$ with $p = -1$ in Figure 4.6 is the basic annular pair $\{s_{-3}, s_2\}$.

(2) $\{s_2, c_3^{-5}\}$ with $p = -2$ in Figure 4.8 is the basic annular pair $\{s_{-3}, s_2\}$.

Proof of Lemma 4.5. Although we can depict the isotopies showing the lemma, we give a proof using a more general argument.

(1) We show that the exterior $X = S^3 - \text{int}N(s_{-3} \cup c_2^{-5} \cup T_{-3,2})$ is homeomorphic to $(\text{the twice punctured disk}) \times S^1$. Then, by [3] $s_{-3} \cup c_2^{-5} \cup T_{-3,2}$ is a union of fibers of some Seifert fibration of S^3 . This implies the desired result.

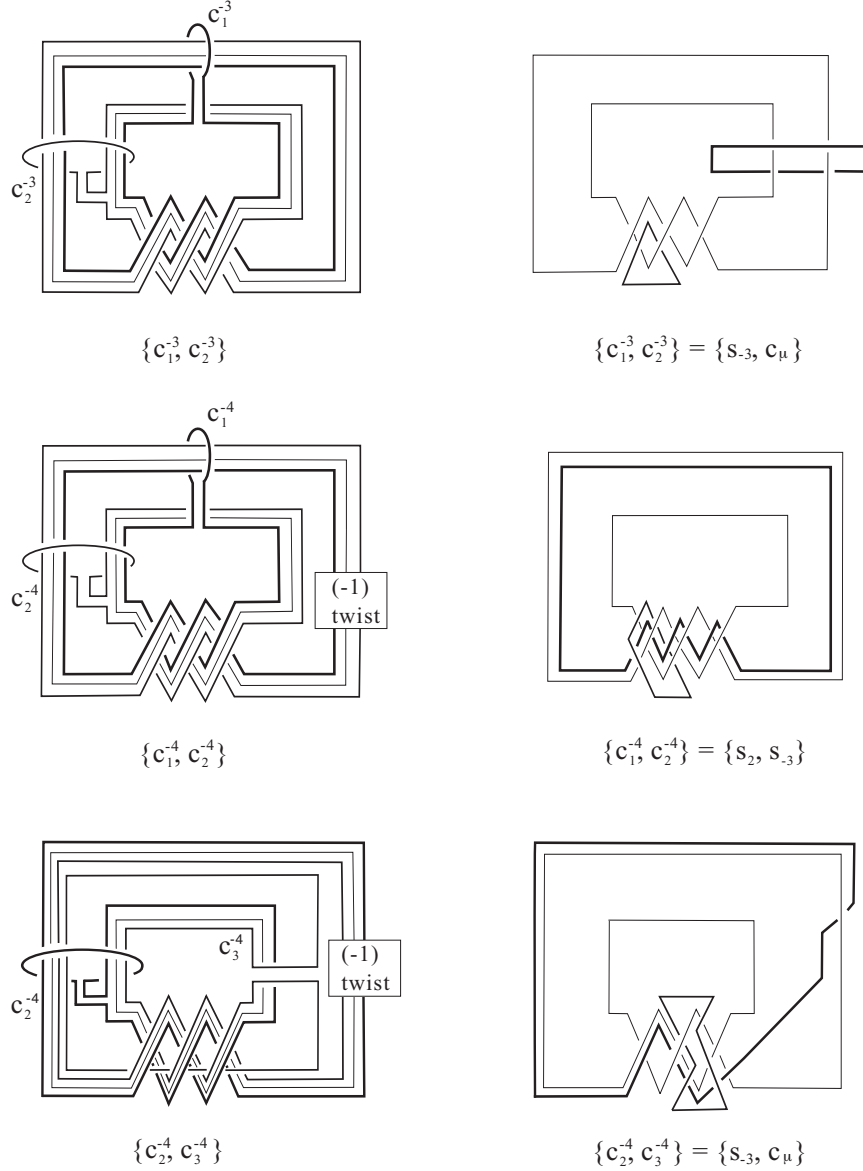


FIGURE 4.9. $\{c_1^{-3}, c_2^{-3}\}$, $\{c_1^{-4}, c_2^{-4}\}$ and $\{c_2^{-4}, c_3^{-4}\}$ are basic annular pairs for $T_{-3,2}$.

Recall that c_2^m is isotopic in $S^3 - \text{int}N(s_{-3} \cup T_{-3,2})$ to a band sum of a simple closed curve in $\partial N(s_{-3})$ with slope p and one in $\partial N(T_{-3,2})$ with slope $m = -5$; let b be the band used in this band sum, where $b \subset S^3 - \text{int}N(s_{-3} \cup T_{-3,2})$. Then s_{-3}, c_2^m and $T_{-3,2}$ cobound an obvious planar surface. By restricting this surface in $X = S^3 - \text{int}N(s_{-3} \cup c_2^m \cup T_{-3,2})$, we obtain a twice punctured disk S properly embedded in X . The boundary slopes of S are $p = -1$ in $\partial N(s_{-3})$, $m = -5$ in $\partial N(T_{-3,2})$, and $m + p + 2\text{lk}(s_{-3}, T_{-3,2}) = -2$ in $\partial N(c_2^{-5})$ by Proposition 2.2. Take a

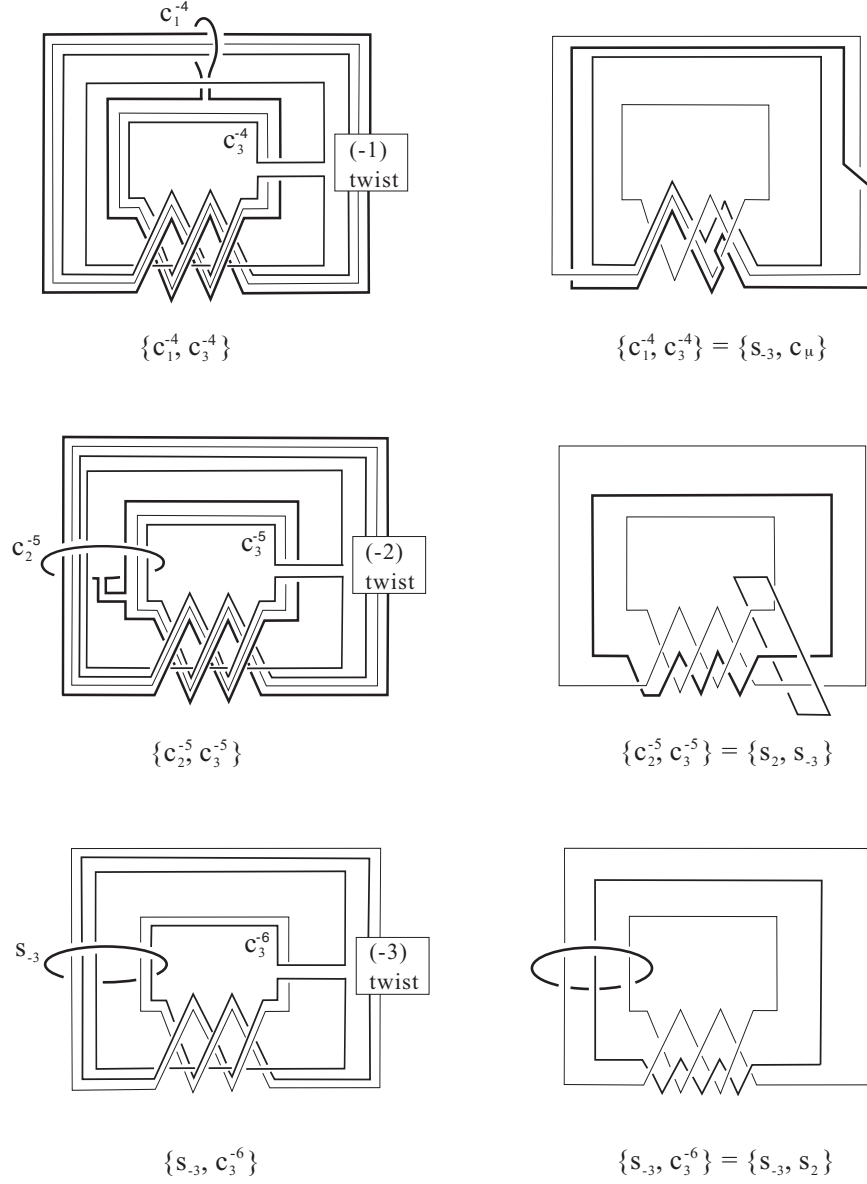


FIGURE 4.10. $\{c_1^{-4}, c_3^{-4}\}$, $\{c_2^{-5}, c_3^{-5}\}$, and $\{s_{-3}, c_3^{-6}\}$ are basic annular pairs for $T_{-3,2}$.

collar neighborhood $S \times I$ of S in X such that $S \times \{0\} = S$ and $\partial S \times I = \partial X \cap S \times I$. Let Y be the closure of $X - S \times I$. Then Y is homeomorphic to the closure of $S^3 - N(s_{-3} \cup T_{-3,2}) - N(a)$, where a is the core of the band b . The arc a is a non-separating arc in an essential annulus in the cable space $S^3 - \text{int}N(s_{-3} \cup T_{-3,2})$ which splits the cable space into a solid torus. This implies that Y is a handlebody of genus 2. We aim to prove $(Y, S) \cong (S \times I, S \times \{0\})$. This implies that X is a fiber

bundle with S a fiber. Since any self-homeomorphism of the twice punctured disk with its boundary setwise invariant is isotopic to the identity, we see $X \cong S \times S^1$.

For simplicity, set $K_1 = s_{-3}$, $K_2 = c_2^{-5}$, $K_3 = T_{-3,2}$, and $\gamma_i = S \cap \partial N(K_i)$, where $i = 1, 2, 3$. We also denote by $[\gamma_i]$ the slope of γ_i in $\partial N(K_i)$. For a set \mathcal{C} of disjoint simple closed curves on a 3-manifold M , $\tau(M; \mathcal{C})$ denote M with 2-handles added along the loops in \mathcal{C} .

Claim 4.6. *For any proper subset \mathcal{C}' of $\{\gamma_1, \gamma_2, \gamma_3\}$, $\tau(Y; \mathcal{C}')$ is a handlebody.*

By [11, Theorem 2] Claim 4.6 implies $(Y, S) \cong (S \times I, S \times \{0\})$ as desired. We prove Claim 4.6 by relating $\tau(Y; \{\gamma_i\})$ with the Dehn surgery $K_i([\gamma_i])$.

Claim 4.7. $\tau(Y, \{\gamma_i\}) \cong K_i([\gamma_i]) - \text{int}N(K_j)$, where $i \neq j$.

Proof of Claim 4.7. Let V be the filled solid torus in $K_i([\gamma_i])$. Cut V by disks bounded by two meridians $S \times \{0, 1\} \cap N(K_i)$ into two 3-balls W_1, W_2 . We may assume $W_1 \cap Y = \partial N(K_i) \cap Y$, so that W_1 (resp. W_2) is attached to Y (resp. $S \times I$) as a 2-handle. Then, for $\{i, \alpha, \beta\} = \{1, 2, 3\}$, $K_i([\gamma_i]) = X \cup V \cup N(K_\alpha) \cup N(K_\beta) = (Y \cup W_1) \cup (S \times I \cup W_2) \cup N(K_\alpha) \cup N(K_\beta)$. Note $Y \cup W_1 \cong \tau(Y; \{\gamma_i\})$. In $K_i([\gamma_i])$, S with γ_i capped off by a meridian disk of V is an annulus connecting $\gamma_\alpha \subset N(K_\alpha)$ and $\gamma_\beta \subset N(K_\beta)$. Hence $(S \times I \cup W_2) \cup N(K_\alpha) \cup N(K_\beta)$ is a solid torus isotopic in $K_i([\gamma_i])$ to both $N(K_\alpha)$ and $N(K_\beta)$. We then see that $\tau(Y; \{\gamma_i\}) \cong K_i([\gamma_i]) - \text{int}N(K_j)$, where $j \in \{\alpha, \beta\}$. \square (Claim 4.7)

Since $s_{-3} \cup c_2^{-5}$ is a Hopf link, $s_{-3}([\gamma_1]) - \text{int}N(c_2^{-5})$ and $c_2^{-5}([\gamma_2]) - \text{int}N(s_{-3})$ are solid tori. The manifold $T_{-3,2}([\gamma_3]) - \text{int}N(s_{-3})$ is obtained from the cable space $S^3 - \text{int}N(s_{-3} \cup T_{-3,2})$ by Dehn-filling $\partial N(T_{-3,2})$ along γ_3 . Since $[\gamma_3] = -5$, γ_3 in $\partial N(T_{-3,2})$ meets a fiber of a Seifert fibration of the cable space exactly once. Hence, $T_{-3,2}([\gamma_3]) - \text{int}N(s_{-3})$ is a solid torus, so that $\tau(Y, \{\gamma_i\})$ is a solid torus for any i .

Claim 4.8. *For a pair $\{\alpha, \beta\} \subset \{1, 2, 3\}$, let $M_{\alpha\beta}$ be the manifold obtained from S^3 by applying $[\gamma_\alpha]$ -surgery on K_α and $[\gamma_\beta]$ -surgery on K_β . Then, $\tau(Y; \{\gamma_\alpha, \gamma_\beta\})$ is homeomorphic to a punctured $M_{\alpha\beta}$.*

Proof of Claim 4.8. As in the proof of Claim 4.7, the filled solid tori in $M_{\alpha\beta}$ are decomposed into 3-balls which are 2-handles attached to Y and $S \times I$. Then, we see that $M_{\alpha\beta} = M_1 \cup M_2 \cup N(K_i)$, where $i \notin \{\alpha, \beta\}$, M_1 (resp. M_2) is Y (resp. $S \times I$) with two 2-handles added along the two annuli $Y \cap \partial N(K_\alpha \cup K_\beta)$ (resp. $S \times I \cap \partial N(K_\alpha \cup K_\beta)$). Note $M_1 \cong \tau(Y; \{\gamma_\alpha, \gamma_\beta\})$ and $M_2 \cong \tau(S \times I; \{\gamma_\alpha, \gamma_\beta\})$. Since M_2 is a 3-ball and attached to $N(K_i)$ along γ_i as a 2-handle, $M_2 \cup N(K_i)$ is a 3-ball. Hence Claim 4.8 follows. \square (Claim 4.8)

Since $N(K_\alpha)$ is isotopic in $K_i([\gamma_i])$ to $N(K_\beta)$ with γ_α sent to γ_β , we see $M_{\alpha\beta} \cong M_{12}$. Since $s_{-3} \cup c_2^{-5}$ is a Hopf link, M_{12} , which is the result of (-1) -surgery on s_{-3} and (-2) -surgery on c_2^{-5} , is the 3-sphere. It then follows from Claim 4.8 that $\tau(Y; \{\gamma_\alpha, \gamma_\beta\})$ is a 3-ball. This completes the proof of Claim 4.6. \square (Claim 4.6)

(2) The arguments in (1) apply after some replacement. There is a twice punctured disk S properly embedded in $X = S^3 - \text{int}N(s_2 \cup c_3^{-5} \cup T_{-3,2})$ such that the boundary slopes of S are $p = -2$ in $\partial N(s_2)$, $m = -5$ in $\partial N(T_{-3,2})$, and $m + p + 2\text{lk}(s_2, T_{-3,2}) = -1$ in $N(c_3^{-5})$. Note also that $s_2 \cup c_3^{-5}$ is a Hopf link. The

arguments after Claim 4.6 hold with s_{-3} and c_2^{-5} replaced by c_3^{-5} and s_2 , respectively. \square (Lemma 4.5)

Theorem 3.24 in [5] shows that under some conditions an annular pair of seiferters is either hyperbolic or basic. Using this theorem and Proposition 4.4, we give a sufficient condition for annular pairs in Figures 4.1–4.4 to be hyperbolic.

Corollary 4.9. (1) *Let m be an integer other than $-5, -6, -7$. Then, an annular pair of seiferters in Figures 4.1–4.4 for $(T_{-3,2}, m)$ is hyperbolic if and only if it is not $\{c_1^{-3}, c_2^{-3}\}$ in Figure 4.4 with $m = -3$ or $\{c_1^{-4}, c_2^{-4}\}$, $\{c_1^{-4}, c_3^{-4}\}$, $\{c_2^{-4}, c_3^{-4}\}$ in Figure 4.4 with $m = -4$.*
 (2) *If $m \leq -8$ or $-1 \leq m$, then nine annular pairs of seiferters for $(T_{-3,2}, m)$ in Figures 4.1–4.4 are all hyperbolic.*

Proof of Corollary 4.9. (1) Let $\{\alpha_m, \beta_m\}$ be an annular pair of seiferters for $(T_{-3,2}, m)$ in Figures 4.1–4.4. Since $m \neq -6$, $T_{-3,2}(m)$ is not a connected sum of lens spaces, so that $(T_{-3,2}, m)$ is a Seifert fibered surgery. The assumption $m \notin \{-5, -7\}$ implies that $T_{-3,2}(m)$ is not a lens space. Since $\alpha_m \cup \beta_m$ is isotopic in $T_{-3,2}(m)$ to a basic annular pair for $T_{-3,2}$, α_m and β_m are exceptional fibers in a Seifert fibration of $T_{-3,2}(m)$. Then, Theorem 3.24 in [5] shows that $\{\alpha_m, \beta_m\}$ is either hyperbolic or basic. Now assertion (1) follows from Proposition 4.4.

(2) Since $m \leq -8$ or $-1 \leq m$, we can apply assertion (1). Then, since $m \notin \{-3, -4\}$, the desired result follows. \square (Corollary 4.9)

Remark 4.10. Corollary 4.9 shows that $(T_{-3,2}, -4)$ has six hyperbolic, annular pairs of seiferters $\{c_\mu, c_2^{-4}\}$, $\{c_\mu, c_3^{-4}\}$, $\{s_{-3}, c_1^{-4}\}$, $\{s_{-3}, c_3^{-4}\}$, $\{s_2, c_1^{-4}\}$, $\{s_2, c_2^{-4}\}$. Note that each of these consists of basic seiferters for $T_{-3,2}$ by Proposition 3.7.

5. STRONGLY INVERTIBLE KNOTS THAT DO NOT ARISE FROM THE PRIMITIVE/SEIFERT-FIBERED CONSTRUCTION

We first review the definition of primitive/Seifert-fibered construction introduced by Dean [4]. Let K be a knot in a genus 2 Heegaard surface F of $S^3 = H \cup_F H'$, where H, H' are genus 2 handlebodies. The surface slope $\gamma_{K,F}$ of $K(\subset F)$ is the isotopy class in $\partial N(K)$ represented by a component of $\partial N(K) \cap F$ which is parallel to K . We denote by $H[K]$ (resp. $H'[K]$) the 3-manifold H (resp. H') with a 2-handle added along K . Note that the surgered manifold $K(\gamma_{K,F})$ is the union of $H[K]$ and $H'[K]$. The knot K is said to be *primitive/Seifert-fibered* with respect to F if $H[K]$ is a solid torus and $H'[K]$ is a Seifert fiber space over $D^2(p, q)$, where $p, q \geq 2$. If K is primitive/Seifert-fibered, then $K(\gamma_{K,F})$ is a lens space, a small Seifert fiber space, or a connected sum of two lens spaces; $(K, \gamma_{K,F})$ is a Seifert surgery. We say that a Seifert surgery (K, m) arises from the *primitive/Seifert-fibered construction* if K is isotopic to a knot L in a genus 2 Heegaard surface F of S^3 in such a way that L is primitive/Seifert-fibered with respect to F and the surface slope $\gamma_{L,F}$ coincides with m . This construction of Seifert surgeries is a modification of Berge's primitive/primitive construction [1] of lens space surgeries. Primitive/primitive knots and primitive/Seifert-fibered knots have tunnel number one, and thus are strongly invertible [4].

Although all known lens space surgeries arise from primitive/primitive constructions, there are infinite families of small Seifert fibered surgeries none of which arises

from the primitive/Seifert-fibered construction [15, 5, 22]. The simplest example is the 1-surgery on the $(-3, 3, 5)$ pretzel knot [15]. For any (K, m) in the families found in [15, 5, 22], K is not strongly invertible. It is natural to raise the following question, and Song [21] gives an example.

Question 5.1. Does there exist a small Seifert fibered surgery on a strongly invertible knot which does not arise from the primitive/Seifert-fibered construction?

Example 5.2 ([21]). (-1) -surgery on the strongly invertible pretzel knot $P(3, -3, -3)$ is a small Seifert fibered surgery which does not arise from the primitive/Seifert-fibered construction.

By twisting $(P(3, -3, -3), -1)$ along a seifert, we extend Song's example to a one-parameter family of Seifert surgeries which give an affirmative answer to Question 5.1. We first show that Song's example is obtained by twisting a Seifert surgery on a trefoil knot.

Lemma 5.3. *Let c be the trivial knot given in Figure 5.1. Then the following hold.*

- (1) *The trivial knot c is a hyperbolic seifert for the Seifert surgery $(T_{-3,2}, -1)$.*
- (2) *(-1) -twist along c converts $(T_{-3,2}, -1)$ to the Seifert surgery $(P(3, -3, -3), -1)$.*
- (3) *The seifert c cannot be obtained from any basic seifert for $T_{-3,2}$ by a single (-1) -move.*

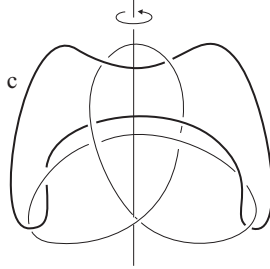


FIGURE 5.1. Seifert c for $(T_{-3,2}, -1)$.

Proof of Lemma 5.3. (1) As shown in Figure 3.3, the trivial knot c_2^{-1} in Figure 5.2 is a seifert for $(T_{-3,2}, -1)$, and an exceptional fiber of index 3 in $T_{-3,2}(-1)$. Let us take the band b as in Figure 5.2, which connects c_2^{-1} and a simple closed curve α_{-1} on $\partial N(T_{-3,2})$ with slope -1 . Isotope $c_2^{-1} \cup b$ as described in Figures 5.3, 5.4. Then Figure 5.5 shows that the (-1) -move via the band b converts $T_{-3,2} \cup c_2^{-1}$ to $T_{-3,2} \cup c$. It follows that c is a seifert for $(T_{-3,2}, -1)$ and an exceptional fiber of index 3 in $T_{-3,2}(-1)$. Then, by [5, Corollary 3.15] c is a hyperbolic seifert or a basic seifert for $(T_{-3,2}, -1)$. However, c is not a basic seifert for $T_{-3,2}$ because $\text{lk}(T_{-3,2}, c) = 0$. This establishes assertion (1). Note that the seifert c is obtained from the basic seifert s_{-3} by applying (-1) -moves twice.

(2) Isotope the link $T_{-3,2} \cup c$ as in Figures 5.6 and 5.7, and twist $T_{-3,2}$ (-1) -time along the seifert c . We then obtain $P(3, -3, -3)$ as required; see Figure 5.8. Since $\text{lk}(T_{-3,2}, c) = 0$, twists on $(T_{-3,2}, -1)$ along c do not change the surgery coefficient. Hence, (-1) -twist along c converts $(T_{-3,2}, -1)$ to $(P(3, -3, -3), -1)$.

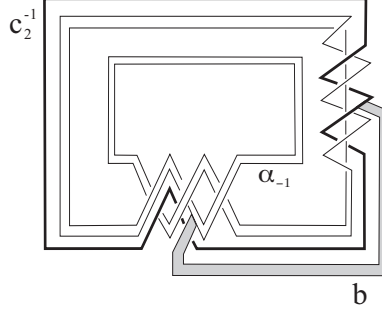


FIGURE 5.2.

(3) Recall that $|\text{lk}(T_{-3,2}, c_\mu)| = 1$, $|\text{lk}(T_{-3,2}, s_{-3})| = 2$, and $|\text{lk}(T_{-3,2}, s_2)| = 3$. If c were obtained from c_μ , s_{-3} , or s_2 by a single (-1) -move, then Proposition 2.2(2) with $\varepsilon = \pm 1$ and $m = -1$ would hold, so that $|\text{lk}(T_{-3,2}, c)| = |1 - \varepsilon|$, $|2 - \varepsilon|$, or $|3 - \varepsilon|$. On the other hand, we have $\text{lk}(T_{-3,2}, c) = 0$. Hence the trivial knot c would be obtained from c_μ by a single (-1) -move. Then, by Remark 3.2 the band b is unique up to isotopy in S^3 , and thus $c = c_\mu \natural_b \alpha_{-1}$ is the same seiferter as c_1^{-1} in Figure 3.2 (c^{-1} in Figure 3.6). However, (-1) -twist on $T_{-3,2}$ along c_1^{-1} gives a trivial knot, and not $P(3, -3, -3)$. This contradicts assertion (2). \square (Lemma 5.3)

Theorem 5.4 below shows that twists along the seiferter c in Figure 5.1 extend Song's example.

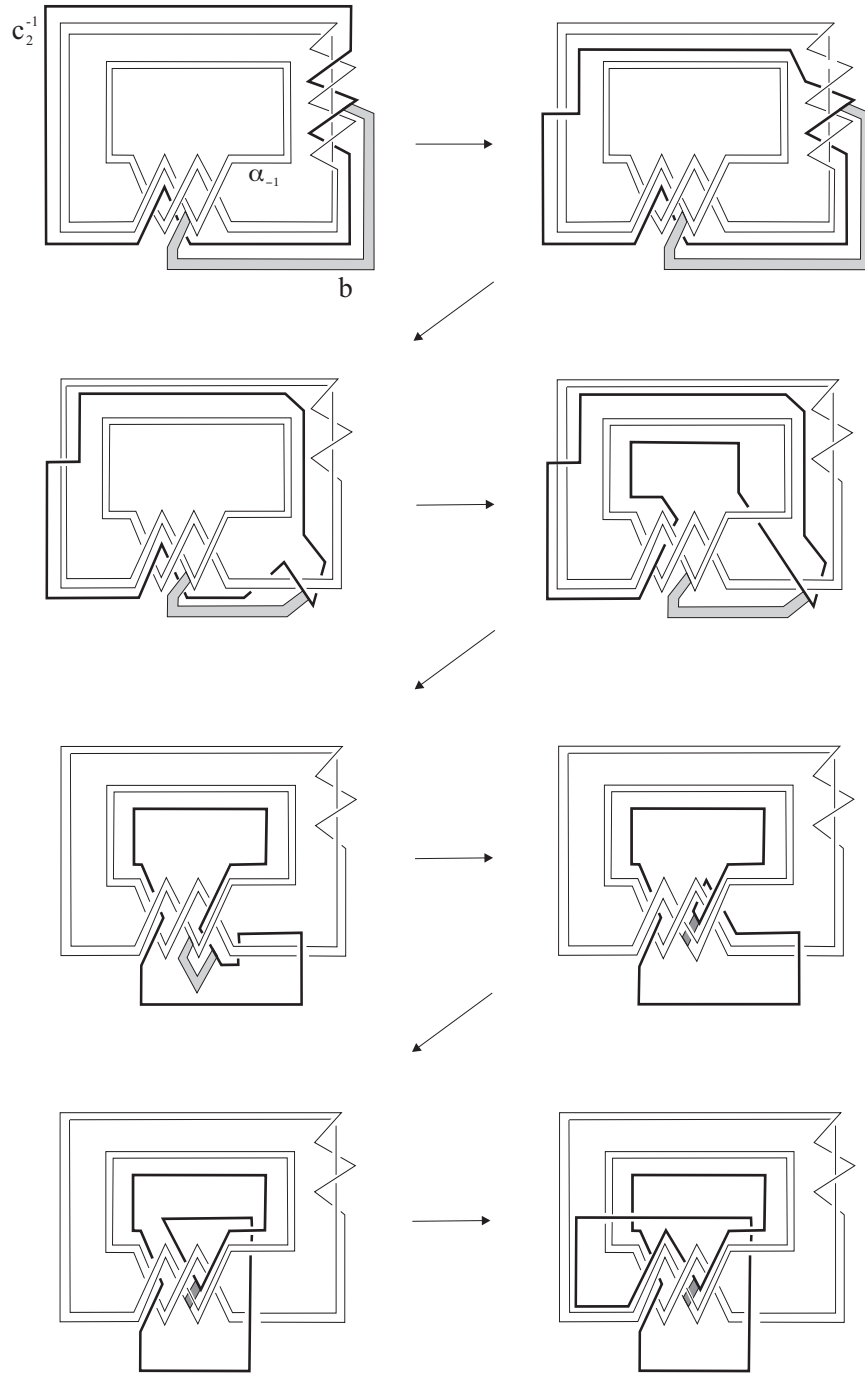
Theorem 5.4. *Let c be the seiferter for $(T_{-3,2}, -1)$ given in Figure 5.1. Then, all Seifert surgeries obtained from $(T_{-3,2}, -1)$ by nontrivial twists along c are small Seifert fibered surgeries on strongly invertible hyperbolic knots. However, none of them arises from the primitive/Seifert-fibered construction. In particular, no knot in this family has tunnel number one.*

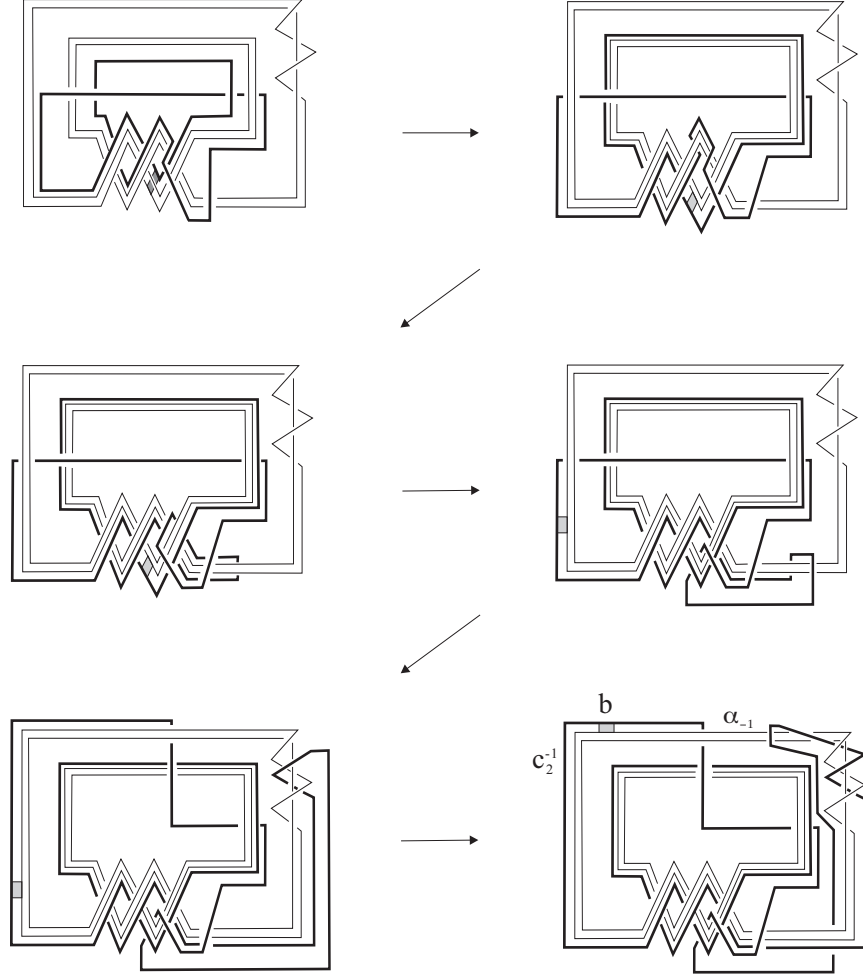
Proof of Theorem 5.4. Let K_p be the knot obtained from $T_{-3,2}$ by twisting p times along c ; then $K_0 = T_{-3,2}$ and $K_{-1} = P(3, -3, -3)$. Note that K_p is the image of $T_{-3,2}$ after performing $(-\frac{1}{p})$ -surgery on the trivial knot c . As shown in Figure 5.1, there is a π -rotation f of S^3 which restricts to inversions of $T_{-3,2}$ and c . Take an f -invariant tubular neighborhood $N(c)$, and extend the involution $f|_{S^3 - \text{int}N(c)}$ to $S^3 = S^3 - \text{int}N(c) \cup_{-\frac{1}{p}} (S^1 \times D^2)$. We obtain a strong inversion of K_p .

Since $\text{lk}(T_{-3,2}, c) = 0$, the surgery slope does not change under the twistings, and thus p -twist along c converts $(T_{-3,2}, -1)$ to $(K_p, -1)$ for any integer p . Let \mathcal{F} be a Seifert fibration of $K_0(-1)$ obtained by extending a Seifert fibration of $S^3 - \text{int}N(K_0)$ in which s_{-3} is an exceptional fiber of index 3; \mathcal{F} is a Seifert fibration over $S^2(2, 3, 5)$. Let μ, λ be a preferred meridian-longitude pair of $N(s_{-3}) \subset S^3$. Then a regular fiber $t(\subset \partial N(s_{-3}))$ of \mathcal{F} is expressed as $[t] = -3[\lambda] + 2[\mu] \in H_1(\partial N(s_{-3}))$.

Lemma 5.5. *$K_p(-1)$ is a small Seifert fiber space over $S^2(2, |10p + 3|, 5)$.*

Proof of Lemma 5.5. The seiferter c_2^{-1} for $(K_0, -1)$ is a band sum of s_{-3} and a simple closed curve α_{-1} in $\partial N(K_0)$ with slope -1 up to isotopy in $S^3 - \text{int}N(K_0)$.

FIGURE 5.3. Isotopy of $T_{-3,2} \cup c_2^{-1}$

FIGURE 5.4. Isotopy of $T_{-3,2} \cup c_2^{-1}$; continued from Figure 5.3

Since $\text{lk}(T_{-3,2}, s_{-3}) = 2$, by Proposition 2.2(3) 0-framing of $N(s_{-3})$ becomes 3-framing of $N(c_2^{-1})$ after isotoping s_{-3} to c_2^{-1} in $K_0(-1)$. Since the seifert c for $(K_0, -1)$ is isotopic in $S^3 - \text{int}N(K_0)$ to a band sum of c_2^{-1} and α_{-1} , and $\text{lk}(T_{-3,2}, c_2^{-1}) = 1$, again by Proposition 2.2(3) the 3-framing of $N(c_2^{-1})$ becomes 4-framing of $N(c)$ after isotoping c_2^{-1} to c in $K_0(-1)$. Hence, a preferred meridian-longitude pair of $N(c)$ is sent to $\mu, \lambda - 4\mu$ curves on $\partial N(s_{-3})$ after isotopy in $K_0(-1)$, where μ, λ is a preferred meridian-longitude pair of $N(s_{-3})$. The $(-\frac{1}{p})$ -surgery slope on $\partial N(c)$ is then sent to $\mu - p(\lambda - 4\mu) = (4p + 1)\mu - p\lambda$ curve on $\partial N(s_{-3})$. Hence, the index of c in the Seifert fibration of $K_p(-1)$ induced from \mathcal{F} is $|((4p + 1)[\mu] - p[\lambda]) \cdot (2[\mu] - 3[\lambda])| = |10p + 3|$. It follows that $K_p(-1)$ is a small Seifert fiber space over $S^2(2, |10p + 3|, 5)$ as desired. \square (Lemma 5.5)

Lemma 5.6. K_p is a hyperbolic knot of genus one for $p \neq 0$.

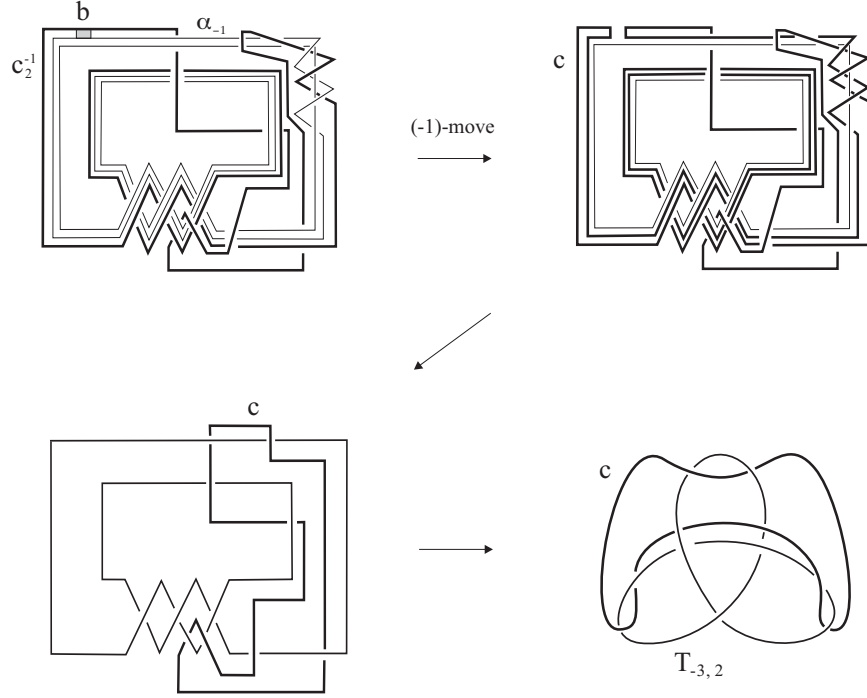
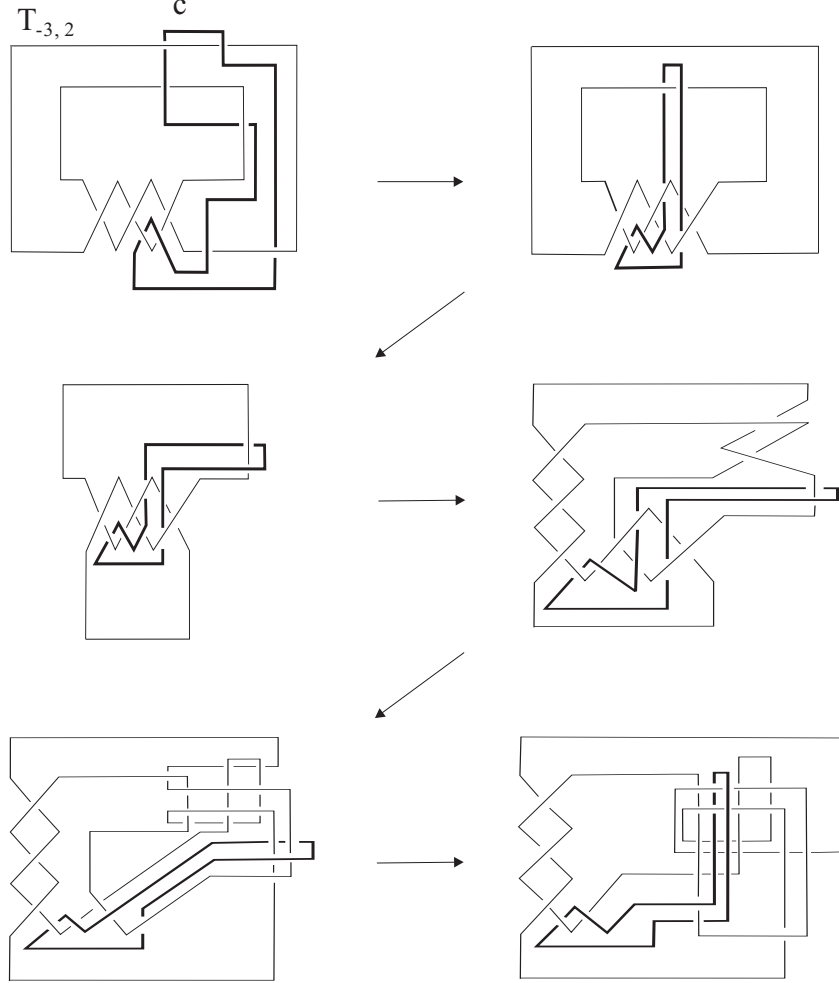


FIGURE 5.5.

Proof of Lemma 5.6. The pretzel knot $P(3, -3, -3) = K_{-1}$ in Figure 5.8 bounds an obvious Seifert surface of genus one disjoint from c . Hence, after $(p+1)$ -twist along c the resulting knot K_p bounds a Seifert surface of genus one for any integer p . Since Lemma 5.5 above implies that K_p is a nontrivial knot, K_p is a knot of genus one.

If K_p is not a hyperbolic knot, then K_p is either a satellite knot or a torus knot. The fact that the genus of K_p is one implies that K_p is a trefoil knot or a satellite knot such that K_p is null-homologous in its companion solid torus V . We see from Lemma 5.5 that K_p is not a trefoil knot for $p \neq 0$, so that K_p is a satellite knot. Since $K_p(-1)$ is a small Seifert fiber space with the trivial first homology group, it does not contain an essential torus [14, Example VI.13]. Hence, the proof of [16, Theorem 1.4] implies that the manifold obtained from V by (-1) -surgery along K_p is a solid torus. However, this is impossible because the winding number of K_p in V is zero [9]. It follows that K_p is a hyperbolic knot for $p \neq 0$. \square (Lemma 5.6)

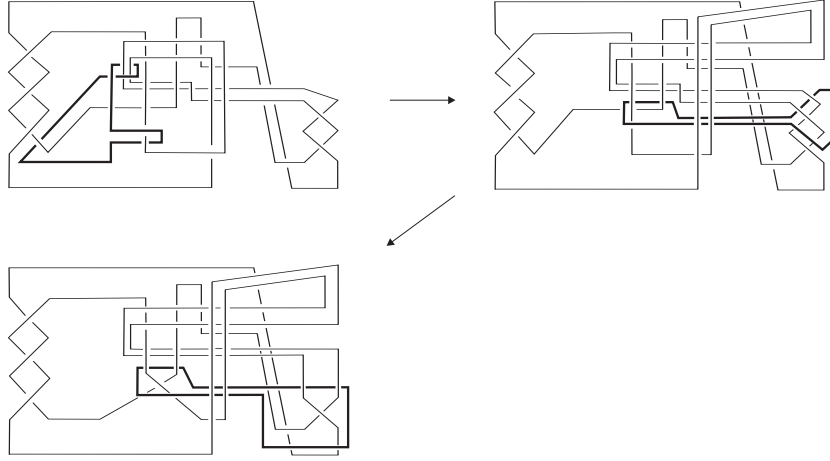
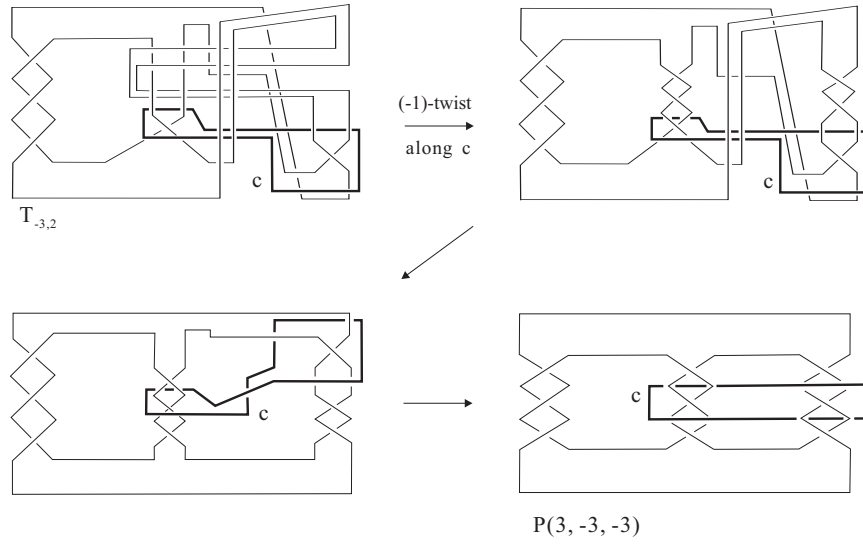
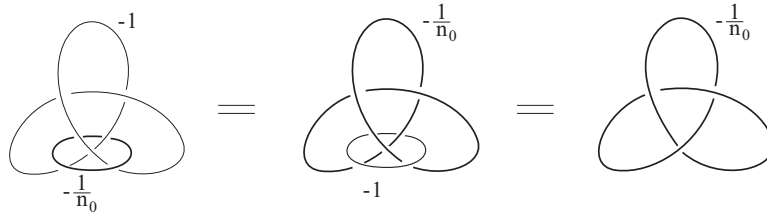
Now suppose that $(K_{p_0}, -1)$ arises from a primitive/Seifert-fibered construction for some p_0 . Then K_{p_0} has tunnel number one. Scharlemann [20] has proved that knots with both tunnel number and genus one are 2-bridge knots or satellite knots, as Goda and Teragaito [10] conjectured. Since 2-bridge knots with Seifert surgeries are twist knots [2], K_{p_0} is a twist knot or its mirror image. In fact, K_{p_0} is a twist knot $Tw(n_0)$ for some n_0 because (-1) -surgery on K_{p_0} is a Seifert surgery. Note that (-1) -surgery on $Tw(n_0)$ yields a Seifert fiber space over $S^2(2, 3, |6n_0 - 1|)$; Figure 5.9 gives a pictorial proof of this fact. On the other hand, $K_{p_0}(-1)$ is a

FIGURE 5.6. Isotopy of $T_{-3,2} \cup c$

Seifert fiber space over $S^2(2, |10p_0 + 3|, 5)$ by Lemma 5.5. It follows $|10p_0 + 3| = 3$; then $p_0 = 0$ as desired. \square (Theorem 5.4)

In [7], using the Montesinos trick, we find an infinite family of small Seifert fibered surgeries on strongly invertible hyperbolic knots which do not arise from the primitive/Seifert-fibered construction. The same family is also obtained from $(T_{-3,2}, m)$ by twisting along the annular pair $\{s_{-3}, c_1^m\}$ in Figure 4.2.

Acknowledgments. The authors would like to thank Toshiyuki Oikawa and Masakazu Teragaito for useful discussions. The third author would like to thank Hyun Jong Song for useful private communications, in particular, letting him know a remarkable example of Seifert surgery $(P(-3, 3, 3), 1)$. Finally, the authors would like to thank the referees for their careful reading and useful comments and, in

FIGURE 5.7. Isotopy of $T_{-3,2} \cup c$; continued from Figure 5.6FIGURE 5.8. (-1) -twist along c converts $T_{-3,2}$ to $P(3, -3, -3)$.FIGURE 5.9. $Tw(n_0)(-1)$ has a surgery description $T_{-3,2}(\frac{-1}{n_0})$.

particular, for pointing out an error in the proof of Proposition 4.4, and would like to thank Mario Eudave-Muñoz for useful comments.

The first and third authors were partially supported by Grants-in-Aid for JSPS Fellows (16-04787). The first author was also partially supported by JSPS Postdoctoral Fellowships for Foreign Researchers at Nihon University (P 04787), COE program “A Base for New Developments of Mathematics into Science and Technology” at University of Tokyo, and INOUE Foundation for Science at Tokyo Institute of Technology. He would like to thank Toshitake Kohno for his hospitality in University of Tokyo and Hitoshi Murakami for his hospitality in Tokyo Institute of Technology. The third author has been partially supported by JSPS Grants-in-Aid for Scientific Research (C) (No.21540098), The Ministry of Education, Culture, Sports, Science and Technology, Japan and Joint Research Grant of Institute of Natural Sciences at Nihon University for 2013.

REFERENCES

- [1] Berge, J.: Some knots with surgeries yielding lens spaces. Unpublished manuscript
- [2] Brittenham, M., Wu, Y.: The classification of Dehn surgeries on 2-bridge knots. *Comm. Anal. Geom.* **9**, 97–113 (2001)
- [3] Burde, G., Murasugi, K.: Links and seifert fiber spaces. *Duke Math. J.* **37**, 89–93 (1970)
- [4] Dean, J.: Small Seifert-fibered Dehn surgery on hyperbolic knots. *Algebr. Geom. Topol.* **3**, 435–472 (2003)
- [5] Deruelle, A., Miyazaki, K., Motegi, K.: Networking Seifert surgeries on knots. *Mem. Amer. Math. Soc.* **217**(1021), viii+130 (2012)
- [6] Deruelle, A., Miyazaki, K., Motegi, K.: Networking Seifert surgeries on knots III. To appear in *Algebr. Geom. Topol.* DOI:10.2140/agt.2014.14.101
- [7] Eudave-Muñoz, M., Jasso, E., Miyazaki, K., Motegi, K.: Seifert fibered surgeries on strongly invertible knots without primitive/seifert positions. To appear in *Topology Appl.*
- [8] Fintushel, R., Stern, R.J.: Constructing lens spaces by surgery on knots. *Math. Z.* **175**, 33–51 (1980)
- [9] Gabai, D.: Surgery on knots in solid tori. *Topology* **28**, 1–6 (1989)
- [10] Goda, H., Teragaito, M.: Tunnel number one genus one non-simple knots. *Tokyo J. Math.* **22**, 99–103 (1999)
- [11] Gordon, C.McA.: On primitive sets of loops in the boundary of a handlebody. *Topology Appl.* **27**, 285–299 (1987)
- [12] Hatcher, A.E.: Notes on basic 3-manifold topology (2000). Freely available at <http://www.math.cornell.edu/hatcher>
- [13] Ishihara, K., Motegi, K.: Band sum operations yielding trivial knots. *Bol. Soc. Mat. Mexicana* (3) **15**, 103–108 (2009)
- [14] Jaco, W.: Lectures on three manifold topology. *CBMS Regional Conference Series in Math.*, vol. 43, Amer. Math. Soc., Providence (1980)
- [15] Mattman, T., Miyazaki, K., Motegi, K.: Seifert fibered surgeries which do not arise from primitive/Seifert-fibered constructions. *Trans. Amer. Math. Soc.* **358**, 4045–4055 (2006)
- [16] Miyazaki, K., Motegi, K.: Seifert fibered manifolds and Dehn surgery. *Topology* **36**, 579–603 (1997)
- [17] Miyazaki, K., Motegi, K.: Seifert fibered manifolds and Dehn surgery II. *Math. Ann.* **311**, 647–664 (1998)
- [18] Motegi, K., Song, H.J.: All integral slopes can be Seifert fibered slopes for hyperbolic knots. *Algebr. Geom. Topol.* **5**, 369–378 (2005)
- [19] Oertel, U.: Closed incompressible surfaces in complements of star links. *Pacific J. Math.* **111**, 209–230 (1984)
- [20] Scharlemann, M.: There are no unexpected tunnel number one knots of genus one. *Trans. Amer. Math. Soc.* **356**, 1385–1442 (2004)
- [21] Song, H.J.: Private communication (2004)

- [22] Teragaito, M.: A Seifert fibered manifold with infinitely many knot-surgery descriptions. Int. Math. Res. Not. **9** (2007), Art. ID rnm 028, 16 pp.

INSTITUTE OF NATURAL SCIENCES, NIHON UNIVERSITY, TOKYO 156-8550, JAPAN
E-mail address: `aderuelle@math.chs.nihon-u.ac.jp`

FACULTY OF ENGINEERING, TOKYO DENKI UNIVERSITY, TOKYO 120-8551, JAPAN
E-mail address: `miyazaki@cck.dendai.ac.jp`

DEPARTMENT OF MATHEMATICS, NIHON UNIVERSITY, TOKYO 156-8550, JAPAN
E-mail address: `motegi@math.chs.nihon-u.ac.jp`